REPRESENTATIONS OF INTEGERS IN TERMS OF GREATEST INTEGER FUNCTIONS AND THE GOLDEN SECTION RATIO

V. E. HOGGATT, JR.
San Jose State University, San Jose, CA 95192
and
MARJORIE BICKNELL-JOHNSON
A. C. Wilcox High School, Santa Clara, CA 95051

Dedicated to ageless George Pólya.

The first and second powers of the golden section ratio, $\alpha = (1+\sqrt{5})/2$, can be used to uniquely represent the positive integers in terms of nested greatest integer functions, relating the compositions of an integer in terms of 1's and 2's with the numbers generated in Wythoff's game. Earlier, Alladi and Hoggatt [1] have shown that there are F_{n+1} compositions of a positive integer n in terms of 1's and 2's, where F_n is the nth Fibonacci number, given by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$. The numbers generated in Wythoff's game have been discussed recently in [2, 3, 8] and by Silber [4].

Suppose we stack greatest integer functions, using α and α^2, to represent the integers in yet another way:

$$1 = [\alpha] = [\alpha[\alpha]] = [\alpha[\alpha[\alpha]]] = [\alpha[\alpha[\alpha[\alpha]]]] = \cdots$$
$$2 = [\alpha^2]$$
$$3 = [\alpha[\alpha^2]]$$
$$4 = [\alpha[\alpha[\alpha^2]]]$$
$$5 = [\alpha^2[\alpha^2]]$$
$$6 = [\alpha[\alpha[\alpha[\alpha^2]]]]$$
$$7 = [\alpha^2[\alpha[\alpha^2]]]$$
$$8 = [\alpha[\alpha^2[\alpha^2]]]$$

Essentially, we start out with the compositions of an integer in terms of 1's and 2's. We put in α^2 wherever there is a 2, and α wherever there is a one, then collapse any strings of α's on the right, since $[\alpha] = 1$. For example, we write the compositions of 5 and 6:

COMPOSITIONS OF 5:

1 + 1 + 1 + 1 + 1	$[\alpha[\alpha[\alpha[\alpha[\alpha]]]]] = [\alpha] = 1$
1 + 1 + 1 + 2	$[\alpha[\alpha[\alpha[\alpha^2]]]] = 6$
1 + 2 + 1 + 1	$[\alpha[\alpha^2[\alpha[\alpha]]]] = [\alpha[\alpha^2]] = 3$
1 + 1 + 2 + 1	$[\alpha[\alpha[\alpha^2[\alpha]]]] = [\alpha[\alpha[\alpha^2]]] = 4$
2 + 1 + 1 + 1	$[\alpha^2[\alpha[\alpha[\alpha]]]] = [\alpha^2] = 2$
1 + 2 + 2	$[\alpha[\alpha^2[\alpha^2]]] = 8$
2 + 1 + 2	$[\alpha^2[\alpha[\alpha^2]]] = 7$
2 + 2 + 1	$[\alpha^2[\alpha^2[\alpha]]] = [\alpha^2[\alpha^2]] = 5$

COMPOSITIONS OF 6:

1 + 1 + 1 + 1 + 1 + 1	$[\alpha] = 1$
1 + 1 + 1 + 1 + 2	$[\alpha[\alpha[\alpha[\alpha[\alpha^2]]]]] = 9$
1 + 1 + 1 + 2 + 1	$[\alpha[\alpha[\alpha[\alpha^2]]]] = 6$
1 + 1 + 2 + 1 + 1	$[\alpha[\alpha[\alpha^2]]] = 4$
1 + 2 + 1 + 1 + 1	$[\alpha[\alpha^2]] = 3$
2 + 1 + 1 + 1 + 1	$[\alpha^2] = 2$
1 + 1 + 2 + 2	$[\alpha[\alpha[\alpha^2[\alpha^2]]]] = 12$
1 + 2 + 1 + 2	$[\alpha[\alpha^2[\alpha[\alpha^2]]]] = 11$
2 + 1 + 1 + 2	$[\alpha^2[\alpha[\alpha[\alpha^2]]]] = 10$
1 + 2 + 2 + 1	$[\alpha[\alpha^2[\alpha^2]]] = 8$
2 + 1 + 2 + 1	$[\alpha^2[\alpha[\alpha^2]]] = 7$
2 + 2 + 1 + 1	$[\alpha^2[\alpha^2]] = 5$
2 + 2 + 2	$[\alpha^2[\alpha^2[\alpha^2]]] = 13$

Notice that the F_6 compositions of 5 gave the representations of the integers 1 through 8, and those of 6, the integers 1 through $F_7 = 13$. We need to systematize; let us arrange the compositions of 5 and 6 so that the representations using α and α^2 are in natural order.

COMPOSITIONS OF 5:	REPRESENTATION:
1 + 1 + 1 + 1 + 1	$[\alpha] = 1$
2 + 1 + 1 + 1	$[\alpha^2] = 2$
1 + 2 + 1 + 1	$[\alpha[\alpha^2]] = 3$
1 + 1 + 2 + 1	$[\alpha[\alpha[\alpha^2]]] = 4$
2 + 2 + 1	$[\alpha^2[\alpha^2]] = 5$
1 + 1 + 1 + 2	$[\alpha[\alpha[\alpha[\alpha^2]]]] = 6$
2 + 1 + 2	$[\alpha^2[\alpha[\alpha^2]]] = 7$
1 + 2 + 2	$[\alpha[\alpha^2[\alpha^2]]] = 8$

COMPOSITIONS OF 6:	REPRESENTATION:
1 + 1 + 1 + 1 + 1 + 1	$[\alpha] = 1$
2 + 1 + 1 + 1 + 1	$[\alpha^2] = 2$
1 + 2 + 1 + 1 + 1	$[\alpha[\alpha^2]] = 3$
1 + 1 + 2 + 1 + 1	$[\alpha[\alpha[\alpha^2]]] = 4$
2 + 2 + 1 + 1	$[\alpha^2[\alpha^2]] = 5$
1 + 1 + 1 + 2 + 1	$[\alpha[\alpha[\alpha[\alpha^2]]]] = 6$
2 + 1 + 2 + 1	$[\alpha^2[\alpha[\alpha^2]]] = 7$
1 + 2 + 2 + 1	$[\alpha[\alpha^2[\alpha^2]]] = 8$
1 + 1 + 1 + 1 + 2	$[\alpha[\alpha[\alpha[\alpha[\alpha^2]]]]] = 9$
2 + 1 + 1 + 2	$[\alpha^2[\alpha[\alpha[\alpha^2]]]] = 10$
1 + 2 + 1 + 2	$[\alpha[\alpha^2[\alpha[\alpha^2]]]] = 11$
1 + 1 + 2 + 2	$[\alpha[\alpha[\alpha^2[\alpha^2]]]] = 12$
2 + 2 + 2	$[\alpha^2[\alpha^2[\alpha^2]]] = 13$

Notice that the representations of the first eight integers using the compositions of 6 agree with the representations using the compositions of 5.

Theorem 1: Any positive integer n can be represented uniquely in terms of nested greatest integer functions of α and α^2, where the exponents match the order of 1's and 2's in a composition in terms of 1's and 2's of an integer k, $n \leq F_{k+1}$, where any α's appearing to the right of the last appearing α^2 are truncated.

Proof: Arrange all of the F_{k+1} compositions of k so that when α and α^2 are inserted in the method described, then the results are in natural order. Do the same for the F_{k+2} compositions of $(k+1)$ in terms of 1's and 2's. Notice that the representations agree with the first F_{k+1} from k. Now, for the compositions of k, tack on the right side α^2, on the far right of the nested greatest integer functions, and suppress all the excess right α's. This yields, with the new addition, representation for the numbers

$$F_{k+1} + 1,\ F_{k+1} + 2,\ \ldots,\ F_{k+1} + F_k = F_{k+2}.$$

Thus, the process may be continued by mathematical induction. The uniqueness also follows as it was part of the inductive hypothesis and carries through. Theorem 1 is proved more formally as Theorems 5 and 6 in what follows.

Next, we write two lemmas.

Lemma 1: $[\alpha F_n] = F_{n+1}$, n odd, $n \geq 2$;

$[\alpha F_n] = F_{n+1} - 1$, n even, $n \geq 2$.

Proof: From Hoggatt [5, p. 34], for $\beta = (1 - \sqrt{5})/2$,

$$\alpha F_n = F_{n+1} - \beta^n;$$
$$[\alpha F_n] = [F_{n+1} - \beta^n].$$

Since $|\beta^n| < 1/2$, $n \geq 2$, if n is odd, then $\beta^n < 0$, and $[F_{n+1} - \beta^n] = F_{n+1}$, while if n is even, $\beta^n > 0$, making $[F_{n+1} - \beta^n] = F_{n+1} - 1$.

Lemma 2: $[\alpha^2 F_n] = F_{n+2}$, n odd, $n \geq 2$;

$[\alpha^2 F_n] = F_{n+2} - 1$, n even, $n \geq 2$.

Proof: Since $\alpha F_n = F_{n+1} - \beta^n$,

$$\begin{aligned} \alpha^2 F_n &= \alpha F_{n+1} - \alpha\beta^n \\ &= (F_{n+2} - \beta^{n+1}) - \alpha\beta^n \\ &= F_{n+2} - \beta^n(\alpha + \beta) \\ &= F_{n+2} - \beta^n. \end{aligned}$$

Then, $[\alpha^2 F_n] = [F_{n+2} - \beta^n]$ is calculated as in Lemma 1.

Lemma 3: For all integers $k \geq 2$ and $n \geq k$,

$[\alpha^k F_n] = F_{n+k}$ if n is odd;

$[\alpha^k F_n] = F_{n+k} - 1$ if n is even.

Proof: $$\alpha^k F_n = \frac{\alpha^k(\alpha^n - \beta^n)}{\sqrt{5}} - \frac{\beta^{n+k}}{\sqrt{5}} + \frac{\beta^{n+k}}{\sqrt{5}} = \frac{\alpha^{n+k} - \beta^{n+k}}{\sqrt{5}} - \frac{\beta^n(\alpha^k - \beta^k)}{\sqrt{5}}$$

$$= F_{k+n} - \beta^n F_k.$$

Now, $|\beta|^n F_k < 1$ if and only if $|\beta|^n < 1/F_k$, which occurs whenever $n \geq k, k \geq 2$, since

$$\frac{1}{F_k} = \frac{\sqrt{5}}{\alpha^k - \beta^k} = \frac{\sqrt{5}}{(-1/\beta)^k - \beta^k} = \frac{\sqrt{5}\beta^k}{(-1)^k - \beta^{2k}}.$$

If k is even, $k \geq 2$, $\beta^k > 0$, and

$$\frac{\sqrt{5}}{1 - \beta^{2k}} > 1$$

$$\frac{1}{F_k} = \frac{\sqrt{5}}{1 - \beta^{2k}} \cdot \beta^k > \beta^k$$

Similarly, if k is odd, $k \geq 3$, $\beta^k < 0$, and

$$\frac{\sqrt{5}}{-1 - \beta^{2k}} > -1$$

$$\frac{1}{F_k} = \frac{\sqrt{5}}{-1 - \beta^{2k}} \cdot \beta^k < -\beta^k$$

Thus, $|\beta|^n F_k < 1$, and Lemma 2 follows.

Next, observe the form of Fibonacci numbers written with nested greatest integer functions of α and α^2:

$$F_2 = 2 = [\alpha] \qquad F_6 = 8 = [\alpha[\alpha^2[\alpha^2]]]$$
$$F_3 = 2 = [\alpha^2] \qquad F_7 = 13 = [\alpha^2[\alpha^2[\alpha^2]]]$$
$$F_4 = 3 = [\alpha[\alpha^2]] \qquad F_8 = 21 = [\alpha[\alpha^2[\alpha^2[\alpha^2]]]]$$
$$F_5 = 5 = [\alpha^2[\alpha^2]] \qquad F_9 = 34 = [\alpha^2[\alpha^2[\alpha^2[\alpha^2]]]]$$

Theorem 2: $F_{2n+1} = [\alpha^2[\alpha^2[\alpha^2[\ \cdots\ ]]]]$,

and $F_{2n+2} = [\alpha[\alpha^2[\alpha^2[\alpha^2[\ \cdots\ ]]]]]$,

both containing n nested α^2 factors.

Proof: We have illustrated the theorem for $n = 1, 2, \ldots, 9$. Assume that Theorem 2 holds for all $n \leq k$. By Lemma 1,

$$F_{2k+2} = [\alpha F_{2k+1}] = [\alpha[\alpha^2[\alpha^2[\alpha^2[\ \cdots\ ]]]]]$$

for k nested α^2 factors; by Lemma 2,

$$F_{2k+3} = [\alpha^2 F_{2k+1}] = [\alpha^2[\alpha^2[\alpha^2[\alpha^2[\alpha^2[\ \cdots\ ]]]]]]$$

for $(k + 1)$ nested α^2 factors.

Return once again to the listed compositions of 5 and 6 using 1's and 2's, and let us count the numbers of 1's and 2's used totally, and the number of α's and α^2's appearing in the integers represented. We also add the data acquired by listing the compositions of 1, 2, 3, and 4, which appear in the tables if the 1's on the right are truncated carefully.

n	1's	2's	α's	α^2's	Suppressed α's
1	1	0	1	0	$0 = F_3 - 2$
2	2	1	1	1	$1 = F_4 - 2$
3	5	2	2	2	$3 = F_5 - 2$
4	10	5	4	5	$6 = F_6 - 2$
5	20	10	9	10	$11 = F_7 - 2$
6	38	20	19	20	$19 = F_8 - 2$

Define C_n as the nth term in the first Fibonacci convolution [6], [7] sequence 1, 2, 5, 10, 20, 38, ..., where

$$C_n = \sum_{i=1}^{n} F_i F_{n-i} = \frac{nL_{n+1} + 2F_n}{5}$$

and observe where these numbers appear in our table. Note that L_n is the nth Lucas number defined by $L_1 = 1$, $L_2 = 3$, and $L_{n+2} = L_{n+1} + L_n$.

Theorem 2: Write the compositions of n using 1's and 2's, and represent all integers less than or equal to F_{n+1} in terms of nested greatest integer functions of α and α^2 as in Theorem 1. Then

(i) C_n 1's appear;

(ii) C_{n-1} 2's appear;

(iii) C_{n-1} α^2's appear;

(iv) $F_{n+2} - 2$ α's are truncated;

(v) $(C_n - F_{n+2} + 2)$ α's appear.

Proof: Let the table just given form our inductive basis, since (i) through (v) hold for $n = 1, 2, 3, 4, 5, 6$. Let $t(n)$ and $u(n)$ denote the number of times 2 and 1 respectively appear in a count of all such compositions of n. Then, by the rules of formation,

$$t(n) = t(n-2) + t(n-1) + F_{n-1}$$

since we will add a 2 on the right to each composition of $(n - 2)$, giving $t(n - 2)$ 2's already there, and $F_{n-2+1} = F_{n-1}$ new 2's written, and $t(n - 1)$ 2's from the compositions of $(n - 1)$, each of which will have a 1 added onto the right. Since [6]

$$C_n = F_n + C_{n-1} + C_{n-2}$$

has the same recursion relation and $t(n)$ has the starting values of the table, $t(n) = C_{n-1}$ for positive integers n, establishing (ii).

Similarly for (i),

$$u(n) = u(n-1) + F_n + u(n-2)$$

since 1's are added on the right to the compositions of $(n - 1)$, keeping $u(n - 1)$ 1's already appearing and adding $F_{n-1+1} = F_n$ new 1's, and all 1's in $(n - 2)$ will appear, since those compositions have a 2 added on the right. We can again establish $u(n) = C_n$ by induction.

Obviously, (ii) and (iii) must have the same count. Since the number of α's appearing is the difference of the number of 1's used and the number of α's truncated, we have (v) immediately if we prove (iv). But the number of suppressed α's for k is the number suppressed in the preceding set of compositions of $(k - 1)$, each of which had a 1 added on the right, plus the number of new 1's on the right, or,

$$F_{k+1} - 2 + F_k = F_{k+2} - 2,$$

so if the formula holds for $1, 2, 3, \ldots, k - 1$, then it also holds for k, and the number of suppressed α's for n is $F_{n+2} - 2$ by mathematical induction.

Now, we go on to the numbers a_n and b_n, where (a_n, b_n) is a safe-pair in Wythoff's game [2, 4, 8]. We list the first few values for a_n and b_n, and some needed properties:

n	1	2	3	4	5	6	7	8	9	10
a_n	1	3	4	6	8	9	11	12	14	16
b_n	2	5	7	10	13	15	18	20	23	26

(1) $\quad a_k + k = b_k$

(2) $\quad a_n + b_n = a_{b_n}$

(3) $\quad a_{a_n} + 1 = b_n$

(4) $\quad a_{a_n+1} - a_{a_n} = 2$ and $a_{b_n+1} - a_{b_n} = 1$

(5) $\quad b_{a_n+1} - b_{a_n} = 3$ and $b_{b_n+1} - b_{b_n} = 2$

(6) $\quad a_n = [n\alpha]$ and $b_n = [n\alpha^2]$

We first concentrate on the expressions in (6) for a_n and b_n, using the greatest integer function, and compare to Lemmas 1 and 2. We can write Lemma 4 immediately, by letting $n = F_k$ in (6).

Lemma 4: For all positive integers k,

$$a_{F_{2k}} = F_{2k+1} - 1 \quad\text{and}\quad a_{F_{2k+1}} = F_{2k+2};$$
$$b_{F_{2k}} = F_{2k+2} - 1 \quad\text{and}\quad b_{F_{2k+1}} = F_{2k+3}.$$

Next we show that the integer following F_n is always a member of $\{a_n\}$.

Theorem 3: $F_{n+1} + 1 = a_{F_n+1}$.

Proof: Part I: $n + 1$ is even. Let $F_{n+1} = F_{2k} = a_{F_{2k-1}}$ from Lemma 4. Note well that $F_{2k-1} \in \{b_n\}$, and by (4),

$$a_{F_{2k-1}+1} - a_{F_{2k-1}} = 1,$$

so that

$$a_{F_{2k-1}+1} = a_{F_{2k-1}} + 1 = F_{2k} + 1.$$

Part II: $n + 1$ is odd. Let $F_{n+1} = F_{2k+1} = b_{F_{2k-1}}$ from Lemma 4. From (3), we have

$$b_{F_{2k-1}} + 1 = a_{a_{F_{2k-1}}} + 1 + 1 = a_{a_{F_{2k-1}}+1}$$

since $a_{a_n+1} - a_{a_n} = 2$. Thus,

$$F_{2k+1} + 1 = b_{F_{2k-1}} + 1 = a_{a_{F_{2k-1}}+1} = a_{F_{2k}+1}$$

from $F_{2k} = a_{F_{2k-1}}$. This concludes Part II and the theorem.

Theorem 4: $a_{F_{n+2}+1} + 1 = b_{F_{n+1}+1}$.

Proof: Part I: n is even. Let $F_{n+2} = F_{2k} = a_{F_{2k-1}}$ and $F_{2k-1} = b_{F_{2k-3}}$ by Lemma 4, so that (4) yields

$$a_{F_{2k-1}+1} = a_{F_{2k-1}} + 1.$$

From this we get

$$a_{a_{F_{2k-1}}+1} + 1 = a_{a_{F_{2k-1}+1}} + 1 = b_{F_{2k-1}+1}$$

making use of (3). This concludes Part I.

Part II: n is odd. Let $F_{n+2} = F_{2k+1}$. Using Theorem 3 and (3),

$$a_{F_{2k+1}+1} + 1 = a_{a_{F_{2k}}+1} + 1 = b_{F_{2k}+1},$$

which concludes Part II and the proof of the theorem.

Comments: We have seen that

$$F_{n+2} + 1 = a_{F_{n+1}+1}$$

from Theorem 3, and

$$a_{F_{n+1}+1} + 1 = b_{F_n+1}$$

from Theorem 4. Thus, the sequence of consecutive b_j's,

$$b_{F_n+1}, b_{F_n+2}, b_{F_n+3}, \ldots, b_{F_{n+1}},$$

and consecutive a_j's,

$$a_{F_{n+1}+1}, a_{F_{n+1}+2}, a_{F_{n+1}+3}, \ldots, a_{F_{n+2}},$$

cover the sequence

$$F_{n+2} + 1, F_{n+2} + 2, F_{n+2} + 3, \ldots, F_{n+3},$$

where, if $F_{n+1} = F_{2k+1}$, then $b_{F_{2k-1}} = F_{2k+1} = F_{n+3}$, and if $F_{n+1} = F_{2k}$, then $a_{F_{n+2}} = a_{F_{2k+1}} = F_{2k+2} = F_{n+3}$. The sequences $\{a_n\}$ and $\{b_n\}$ are such that their disjoint union covers the positive integers, and there are F_{n-1} of the b_j's and F_n of the a_j's, or collectively, F_{n+1} all together. The interval $[F_{n+2} + 1, F_{n+3}]$ contains precisely F_{n+1} positive integers. We have shown that the union of the two sequences are precisely the integers on this interval. We now are ready to prove Theorem 5 by mathematical induction.

Theorem 5: If α^2 is added onto the right of the specified function for the compositions of n *properly ordered*, then we obtain the integers

$$F_{n+2} + 1, F_{n+2} + 2, \ldots, F_{n+2} + F_{n+1} = F_{n+3}.$$

Proof: By our previous discussions, Theorem 5 is true for $n = 1, 2, \ldots, 6$. Assume it is true for $n = k - 1$ and $n = k$. Then, let us add α^2 on the left to each value of the specified function, making the result be the F_k successive b_j's

$$b_{F_{k+1}+1}, b_{F_{k+1}+2}, \ldots, b_{F_{k+2}},$$

and let us add α on the left to each value of the specified function, to obtain the F_{k+1} successive a_j's,

$$a_{F_{k+2}+1}, a_{F_{k+2}+2}, \ldots, a_{F_{k+3}}.$$

These numbers together give the interpretation of compositions of $(k + 1)$ with α^2 on the right, so we must get $F_{k+3} + 1, F_{k+3} + 2, \ldots, F_{k+4}$. There are F_k consecutive b_j's and F_{k+1} consecutive a_j's which fit together precisely to cover the above interval by the discussion preceding Theorem 5, giving us a proof by mathematical induction.

Theorem 6: The F_{n+2} compositions of $(n + 1)$ using 1's and 2's when put into the nested greatest integer function with 1 and 2 the exponents on α can be arranged so that the results are the integers 1, 2, ..., F_{n+2} in sequence.

Proof: We have illustrated Theorem 6 for $n = 1, 2, \ldots, 5$. Assume that the F_n compositions for $(n - 1)$ have been so arranged in the nested greatest integer function representations. By Theorem 5, the results of putting 2 on the right of the compositions, or an α^2 on the right of each representation, yields the numbers $F_{n+2} + 1, F_{n+2} + 2, \ldots, F_{n+3}$. The adding of a one to the right of compositions of $(n + 1)$ yields a composition of $(n + 2)$ but it does not change the results of the nested greatest integer representations. Thus the list now goes for compositions of $(n + 2)$, the first F_{n+2} coming from the one added on the right of those for $(n + 1)$ and the F_{n+1} more coming from the two added on the right of those for n. Thus, by mathematical induction, we complete the proof of the theorem for all $n \geq 1$.

The above proof is constructive, as it yields the proper listing of the composition for $(n + 2)$ if we have them for n and for $(n + 1)$.

Notice the pattern of our representations if we simply record them in a different way:

$$1 = [\alpha] = a_1$$
$$2 = [\alpha^2] = b_1$$
$$3 = [\alpha[\alpha^2]] = a_{b_1}$$
$$4 = [\alpha[\alpha[\alpha^2]]] = a_{a_{b_1}}$$
$$5 = [\alpha^2[\alpha^2]] = b_{b_1}$$
$$6 = [\alpha[\alpha[\alpha[\alpha^2]]]] = a_{a_{a_{b_1}}}$$
$$7 = [\alpha^2[\alpha[\alpha^2]]] = b_{a_{b_1}}$$
$$8 = [\alpha[\alpha^2[\alpha^2]]] = a_{b_{b_1}}$$

In other words, Theorems 3 through 6 and Lemma 4 will allow us to write a representation of an integer such that each α in its nested greatest integer function becomes a subscripted a, and each α^2 a subscripted b, in a continued subscript form.

Next, we present a simple scheme for writing the representations of the integers in terms of nested greatest integer functions of α and α^2, as in Theorems 1 and 6. We use the difference of the subscripts of Fibonacci numbers to obtain the exponents 1 and 2, or the compositions of n in terms of 1's and 2's, by using F_{n+1} in the rightmost column. We illustrate for $n = 6$, using F_7. Notice that every other column in the table is the subscript difference of the two adjacent Fibonacci numbers, and compare with the compositions of 6 and the representations of the integers 1, 2, ..., 13 in natural order given just before Theorem 1. We use the Fibonacci numbers as place holders. One first writes the column of 13 F_7's, which is broken into 8 F_6's and 5 F_5's. The 8 F_6's are broken into 5 F_5's and 3 F_4's, and the 5 F_5's into 3 F_4's and 2 F_3's. The pattern continues in each column, until each F_2 is broken into F_1 and F_0, so ending with F_1. In each new column, 1 always replaces F_n F_n's with F_{n-1} F_{n-1}'s and F_{n-2} F_{n-2}'s. Notice that the next level, representing all integers through $F_8 = 21$, would be formed by writing 21 F_8's in the right column, and the present array as the top 13 = F_7 rows, and the array ending in 8 F_6's now in the top 8 = F_6 rows would appear in the bottom eight rows. Notice further that, just as in the proofs of Theorems 1 and 6, this scheme puts a 1 on the right of all compositions of $(n - 1)$ and a 2 on the right of all compositions of $(n - 2)$.

SCHEME TO FORM ARRAY OF COMPOSITIONS OF INTEGERS $n \le F_7$

F_1	1	F_2	1	F_3	1	F_4	1	F_5	1	F_6	1	F_7	$n = 1$
		F_1	2	F_3	1	F_4	1	F_5	1	F_6	1	F_7	$n = 2$
		F_1	1	F_2	2	F_4	1	F_5	1	F_6	1	F_7	$n = 3$
		F_1	1	F_2	1	F_3	2	F_5	1	F_6	1	F_7	$n = 4$
				F_1	2	F_3	2	F_5	1	F_6	1	F_7	$n = 5$
		F_1	1	F_2	1	F_3	1	F_4	2	F_6	1	F_7	$n = 6$
				F_1	2	F_3	1	F_4	2	F_6	1	F_7	$n = 7$
				F_1	1	F_2	2	F_4	2	F_6	1	F_7	$n = 8$
		F_1	1	F_2	1	F_3	1	F_4	1	F_5	2	F_7	$n = 9$
				F_1	2	F_3	1	F_4	1	F_5	2	F_7	$n = 10$
				F_1	1	F_2	2	F_4	1	F_5	2	F_7	$n = 11$
				F_1	1	F_2	1	F_3	2	F_5	2	F_7	$n = 12$
						F_1	2	F_3	2	F_5	2	F_7	$n = 13$

Within the array just given, we have used 8 F_6's, 5 F_5's, 6 F_4's, 6 F_3's, 5 F_2's, and 8 F_1's, where $8 + 5 + 6 + 6 + 5 + 8 = 38 = C_6$, where again C_n is the nth element in the Fibonacci convolution sequence. These coefficients appear in the array:

						row sum
1						1
1	1					2
2	1	2				5
3	2	2	3			10
5	3	4	3	5		20
8	5	6	6	5	8	38
...	...	...	...	...	...	...
F_n	$1F_{n-1}$	$2F_{n-2}$	$3F_{n-3}$	$5F_{n-4}$	$8F_{n-5}$ ...	C_n

The rows give the number of F_n's, F_{n-1}'s, F_{n-2}'s, ..., used in the special array to write the compositions of n in natural order. Properties of the array itself will be considered later.

Now we turn to the Lucas numbers. We observe

$$L_2 = 3 = [\alpha[\alpha^2]]$$
$$L_3 = 4 = [\alpha[\alpha[\alpha^2]]]$$
$$L_4 = 7 = [\alpha^2[\alpha[\alpha^2]]]$$
$$L_5 = 11 = [\alpha[\alpha^2[\alpha[\alpha^2]]]]$$
$$L_6 = 18 = [\alpha^2[\alpha^2[\alpha[\alpha^2]]]]$$
$$L_7 = 29 = [\alpha[\alpha^2[\alpha^2[\alpha[\alpha^2]]]]]$$
$$L_8 = 47 = [\alpha^2[\alpha^2[\alpha^2[\alpha[\alpha^2]]]]]$$

Thus, it appears that $[\alpha L_n] = L_{n+1}$ if n is odd, and that $[\alpha^2 L_n] = L_{n+2}$ if n is even. Also, we see the form of Lucas numbers, and can compare them with the representation of Fibonacci numbers. We first need a lemma.

Lemma 5: $-1 < \beta^n\sqrt{5} < 1$ for $n \geq 2$.

Proof: $\beta^2 = (3 - \sqrt{5})/2$, and $\beta^2\sqrt{5} = (3\sqrt{5} - 5)/2 < .85 < 1$. Thus,

$$0 < \beta^{2n}\sqrt{5} \leq \beta^2\sqrt{5} < 1 \quad \text{for} \quad n \geq 1.$$

If $0 < \beta^2\sqrt{5} < 1$, then $0 > \beta^3\sqrt{5} > -1$, so that

$$-1 < \beta^{2n+1}\sqrt{5} < 0 \quad \text{for} \quad n \geq 1,$$

establishing Lemma 5.

Lemma 6: $[\alpha L_n] = L_{n+1}$ for n even, if $n \geq 2$;

$[\alpha L_n] = L_{n+1} - 1$ for n odd, if $n \geq 3$.

Proof: Apply Lemma 5 to the expansion of αL_n:

$$\begin{aligned}\alpha L_n &= \alpha(\alpha^n + \beta^n) = \alpha^{n+1} + \beta^{n+1} + \alpha\beta^n - \beta^{n+1}\\ &= L_{n+1} + \beta^n(\alpha - \beta) = L_{n+1} + \beta^n\sqrt{5}.\end{aligned}$$

Lemma 7: $[\alpha^2 L_n] = L_{n+2}$ if n is even and $n \geq 2$;

$[\alpha^2 L_n] = L_{n+2} - 1$, if n is odd and $n \geq 1$.

Proof: We apply Lemma 5 to

$$\begin{aligned}\alpha^2 L_n &= \alpha^2(\alpha^n + \beta^n) = \alpha^{n+2} + \beta^{n+2} + \beta^n(\alpha^2 - \beta^2)\\ &= L_{n+2} + \beta^n\sqrt{5}.\end{aligned}$$

Theorem 7: The Lucas numbers L_n are representable uniquely in terms of nested greatest integer functions of α and α^2 in the forms

$$\begin{aligned}L_{2n+1} &= [\alpha[\alpha^2[\alpha^2[\alpha^2 \ldots [\alpha[\alpha^2]] \ldots]]]],\\ L_{2n} &= [\alpha^2[\alpha^2[\alpha^2 \ldots [\alpha[\alpha^2]] \ldots]]],\end{aligned}$$

where the number of α^2 consecutively is $(n - 1)$, $n \geq 1$.

Proof: Theorem 7 has already been illustrated for $n = 1, 2, \ldots, 8$. A proof by mathematical induction follows easily from Lemmas 6 and 7.

Comparing Theorems 2 and 7, we notice that the representations of F_k and L_{k+1} are very similar, with the representation of L_{k+1} duplicating that of F_k with $[\alpha[\alpha^2]]$ added on the far right. We write

Theorem 8:
$$\begin{aligned}F_{2n+2} &= [\alpha[\alpha^2[\alpha^2 \ldots [\alpha^2] \ldots]]] \quad \text{and}\\ L_{2n+3} &= [\alpha[\alpha^2[\alpha^2 \ldots [\alpha^2[\alpha[\alpha^2]]] \ldots]]];\\ F_{2n+1} &= [\alpha^2[\alpha^2[\alpha^2 \ldots [\alpha^2] \ldots]]] \quad \text{and}\\ L_{2n+2} &= [\alpha^2[\alpha^2[\alpha^2 \ldots [\alpha^2[\alpha[\alpha^2]]] \ldots]]],\end{aligned}$$

where there are n consecutive α^2's.

Theorem 8, restated, shows that if a 1 and a 2 is added on the right to the composition of $(k - 1)$ in terms of 1's and 2's that gave rise to F_k, one obtains L_{k+1}. If we add a 1 and a 2 on the right of the compositions of n, we observe:

$n = 1$:	1	1 2	$[\alpha[\alpha[\alpha^2]]] = 4 = F_4 + 1 = L_3$
$n = 2$:	1 1	1 2	$[\alpha[\alpha[\alpha[\alpha^2]]]] = 6 = F_5 + 1$
	2	1 2	$[\alpha^2[\alpha[\alpha^2]]] = 7 = F_5 + 2 = L_4$
$n = 3$:	1 1 1	1 2	$[\alpha[\alpha[\alpha[\alpha[\alpha^2]]]]] = 9 = F_6 + 1$
	2 1	1 2	$[\alpha^2[\alpha[\alpha[\alpha^2]]]] = 10 = F_6 + 2$
	1 2	1 2	$[\alpha[\alpha^2[\alpha[\alpha^2]]]] = 11 = F_6 + 3 = L_5$
$n = 4$:	1 1 1 1	1 2	$[\alpha[\alpha[\alpha[\alpha[\alpha[\alpha^2]]]]]] = 14 = F_7 + 1$
	2 1 1	1 2	$[\alpha^2[\alpha[\alpha[\alpha[\alpha^2]]]]] = 15 = F_7 + 2$
	1 2 1	1 2	$[\alpha[\alpha^2[\alpha[\alpha[\alpha^2]]]]] = 16 = F_7 + 3$
	1 1 2	1 2	$[\alpha[\alpha[\alpha^2[\alpha[\alpha^2]]]]] = 17 = F_7 + 4$
	2 2	1 2	$[\alpha^2[\alpha^2[\alpha[\alpha^2]]]] = 18 = F_7 + 5 = L_6$

Theorem 9: If to the compositions of n in terms of 1's and 2's, written in the order producing representations of 1, 2, ..., F_{n+1} in terms of nested greatest integer functions of α and α^2 in natural order, we add a 1 and a 2 on the right, then the resulting nested greatest integer functions of α and α^2 have values

$$F_{n+3} + 1,\ F_{n+3} + 2,\ \ldots,\ F_{n+3} + F_{n+1} = L_{n+2}.$$

Now, notice that, since the representation giving rise to a Lucas number in the nested greatest integer representation ends with a 1 and a 2, the next representation, taken in natural order, will end in a 2 and a 2. Consider the compositions of n, where we add two 2's on the right:

$n = 1$:	1	2 2	$[\alpha[\alpha^2[\alpha^2]]] = 8 = L_4 + 1 = F_6$
$n = 2$:	1 1	2 2	$[\alpha[\alpha[\alpha^2[\alpha^2]]]] = 12 = L_5 + 1$
	2	2 2	$[\alpha^2[\alpha^2[\alpha^2]]] = 13 = L_5 + 2 = F_7$
$n = 3$:	1 1 1	2 2	$[\alpha[\alpha[\alpha[\alpha^2[\alpha^2]]]]] = 19 = L_6 + 1$
	2 1	2 2	$[\alpha^2[\alpha[\alpha^2[\alpha^2]]]] = 20 = L_6 + 2$
	1 2	2 2	$[\alpha[\alpha^2[\alpha^2[\alpha^2]]]] = 21 = L_6 + 3 = F_8$
$n = 4$:	1 1 1 1	2 2	$[\alpha[\alpha[\alpha[\alpha[\alpha^2[\alpha^2]]]]]] = 30 = L_7 + 1$
	2 1 1	2 2	$[\alpha^2[\alpha[\alpha[\alpha^2[\alpha^2]]]]] = 31 = L_7 + 2$
	1 2 1	2 2	$[\alpha[\alpha^2[\alpha[\alpha^2[\alpha^2]]]]] = 32 = L_7 + 3$
	1 1 2	2 2	$[\alpha[\alpha[\alpha^2[\alpha^2[\alpha^2]]]]] = 33 = L_7 + 4$
	2 2	2 2	$[\alpha^2[\alpha^2[\alpha^2[\alpha^2]]]] = 34 = L_7 + 5 = F_9$

Theorem 10: If to the compositions of n in terms of 1's and 2's, written in the order that produces representations of 1, 2, ..., F_{n+1} in natural order in terms of nested greatest integer functions of α and α^2, we add two 2's on

the right, then the resulting nested greatest integer functions of α and α^2 have the consecutive values

$$L_{n+3} + 1,\ L_{n+3} + 2,\ \ldots,\ L_{n+3} + F_{n+1} = F_{n+5}.$$

We are now in a position to count in two different ways all the α's and α^2's appearing in the display of all integers from 1 through L_n simultaneously. Of the F_{n+1} compositions of n, there are F_n which end in a $\underline{1}$, and F_{n-1} which end in a $\underline{2}$. Those ending in a $\underline{1}$ are the compositions of $(n - 1)$ with our $\underline{1}$ added, while those ending in a $\underline{2}$ are the compositions of $(n - 2)$ with our $\underline{2}$ added. Now, if we add $\underline{2}$ to each of these F_{n+1} compositions, by Theorem 5, we get the numbers

$$F_{n+2} + 1,\ F_{n+2} + 2,\ \ldots,\ F_{n+2} + F_{n+1} = F_{n+3}.$$

Of these, there were F_n ending in a $\underline{1}$, which now end in a $\underline{1\text{-}2}$ and cover the numbers

$$F_{n+2} + 1,\ F_{n+2} + 2,\ \ldots,\ F_{n+2} + F_n = L_{n+1},$$

and those that end in a $\underline{2\text{-}2}$ cover the numbers

$$L_{n+1} + 1,\ L_{n+1} + 2,\ \ldots,\ L_{n+1} + F_{n+1} = F_{n+3}$$

when used in the nested greatest integer functions of α and α^2 in natural order. We can now count the number of α's and α^2's used to display all the representations of the integers from 1 to L_{n+1}. We count all of those up to and including F_{n+3} by Theorem 2, and subtract the total α and α^2 content of the compositions of $(n - 2)$, which is C_{n-2} α's and C_{n-3} α^2's, and subtract $2F_{n-1}$ α^2's, or, we can count all of those up to and including F_{n+2}, and add on the F_n α's and F_n α^2's, and add the number of 1's in the compositions of $(n - 1)$, which all become α's in counting from $F_{n+2} + 1$ through $F_{n+2} + F_n = L_{n+1}$. The first method gives us, for the number of α's,

$$(C_{n+2} - F_{n+4} + 2) - C_{n-2},$$

and for the number of α^2's,

$$C_{n+1} - C_{n+3} - 2F_{n-1}.$$

The second method gives the number of α's as

$$(C_{n+1} - F_{n+3} + 2) + C_{n-2} + F_n,$$

which simplifies to

$$C_{n+1} + C_{n-1} - 2F_{n+1} + 2,$$

and the number of α^2's as

$$C_n + C_{n-2} + F_n,$$

finishing a proof of Theorem 11.

Theorem 11: Write the compositions of $(n + 2)$ using 1's and 2's, and represent all integers less than or equal to L_{n+1} in terms of nested greatest integer functions of α and α^2 in natural order as in Theorem 1. Then,

(i) $(C_{n+2} - C_{n-2} - F_{n+4} + 2) = (C_{n+1} + C_{n-1} - 2F_{n+1} + 2)$

is the number of α's appearing, and

(ii) $(C_{n+1} + C_{n-2} + F_n) = (C_{n+1} - C_{n-3} - 2F_{n-1})$

is the number of α^2's appearing.

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A PRIMER ON STERN'S DIATOMIC SEQUENCE

CHRISTINE GIULI

University of Santa Clara, Santa Clara, CA 95053

and

ROBERT GIULI

University of California, Santa Cruz, CA 96050

PART III: ADDITIONAL RESULTS

An examination of the sequence yields corollaries to some of the previously known results. Being fundamentally Fibonacci minded, and at the onset not aware of the works of Stern, Eisenstein, Lehmer and Lind, we noticed the following results not already mentioned—some may even seem trivial.

(1) $s(n,1) = n$
$s(n,2) = n - 1$
$s(n,4) = n - 2$
$\vdots$
$s(n,2^m) = n - m$

(2) $s(n,a2^m) = s(n - m,a)$

(3) Another statement of symmetry is $s(n,2^{n-2} - a) = s(n,2^{n-2} + a)$

(4) $s(n,2^{n-1}) = 1$
$s(n,2^{n-2}) = 2$
$\vdots$
$s(n,2^{n-k}) = k$