REFERENCE

1. L. Carlitz. "Restricted Compositions." *The Fibonacci Quarterly* 14, No. 3 (1976):254-264.

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CHEBYSHEV AND FERMAT POLYNOMIALS FOR DIAGONAL FUNCTIONS

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INTRODUCTION

Jaiswal [3] and the author [1] examined rising diagonal functions of Chebyshev polynomials of the second and first kinds, respectively. Also, in [2], the author investigated rising and descending functions of a wide class of sequences satisfying certain criteria. Excluded from consideration in [2] were the Chebyshev and Fermat polynomials that did not satisfy the restricting criteria.

The object of this paper is to complete the above articles by studying *descending* diagonal functions for the Chebyshev polynomials in Part I, and *both* rising and descending diagonal functions for the Fermat polynomials in Part II.

Chebyshev polynomials $T_n(x)$ of the second kind are defined by

(1) $$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x) \qquad T_0(x) = 2,\ T_1(x) = 2x \qquad (n \geq 0),$$

while Chebyshev polynomials $U_n(x)$ of the first kind are defined by

(2) $$U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x) \qquad U_0(x) = 1,\ U_1(x) = 2x \qquad (n \geq 0).$$

Often we write $x = \cos\theta$ to obtain trigonometrical sequences.

PART I

DESCENDING DIAGONAL FUNCTIONS FOR $T_n(x)$

From (1), we obtain

(3) $$\begin{cases} T_0(x) = 2 \\ T_1(x) = 2x \\ T_2(x) = 4x^2 - 2 \\ T_3(x) = 8x^3 - 6x \\ T_4(x) = 16x^4 - 16x^2 + 2 \\ T_5(x) = 32x^5 - 40x^3 + 10x \\ T_6(x) = 64x^6 - 96x^4 + 36x^3 - 2 \\ T_7(x) = 128x^7 - 224x^5 + 112x^3 - 14x \\ T_8(x) = 256x^8 - 512x^6 + 320x^4 - 64x^2 + 2 \\ T_9(x) = 512x^9 - 1152x^7 + 864x^5 - 240x^3 + 18x \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Descending diagonal functions of $x, a_i(x)$ $(i = 1, 2, 3, \ldots)$, for $T_n(x)$ are, from (3) [taking $a_0(x) = 0$],

$$(4)\quad \begin{cases} a_1(x) = 2 \\ a_2(x) = 2x - 2 \\ a_3(x) = 4x^2 - 6x + 2 \\ a_4(x) = 8x^3 - 16x^2 + 10x - 2 \\ a_5(x) = 16x^4 - 40x^3 + 36x^2 - 14x + 2 \\ a_6(x) = 32x^5 - 96x^4 + 112x^3 - 64x^2 + 18x - 2 \\ a_7(x) = 64x^6 - 224x^5 + 320x^4 - 240x^3 + 100x^2 - 22x + 2 \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

These yield

$$(5)\quad a_{n+1}(x) = (2x - 1)a_n(x) = (2x - 2)(2x - 1)^{n-1} \quad (n \geq 1).$$

DESCENDING DIAGONAL FUNCTIONS FOR $U_n(x)$

From (2), we obtain

$$(6)\quad \begin{cases} U_0(x) = 1 \\ U_1(x) = 2x \\ U_2(x) = 4x^2 - 1 \\ U_3(x) = 8x^3 - 4x \\ U_4(x) = 16x^4 - 12x^2 + 1 \\ U_5(x) = 32x^5 - 32x^3 + 6x \\ U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1 \\ U_7(x) = 128x^7 - 192x^5 + 80x^3 - 8x \\ U_8(x) = 256x^8 - 448x^6 + 240x^4 - 40x^2 + 1 \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Descending diagonal functions of $x, b_i(x)$ $(i = 1, 2, 3, \ldots)$, for $U_n(x)$ are, from (6) [taking $b_0(x) = 0$],

$$(7)\quad \begin{cases} b_1(x) = 1 \\ b_2(x) = 2x - 1 \\ b_3(x) = 4x^2 - 4x + 1 = (2x - 1)^2 \\ b_4(x) = 8x^3 - 12x^2 + 6x - 1 = (2x - 1)^3 \\ b_5(x) = 16x^4 - 32x^3 + 24x^2 - 8x + 1 = (2x - 1)^4 \\ b_6(x) = 32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1 = (2x - 1)^5 \\ b_7(x) = 64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1 = (2x - 1)^6 \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

These yield

(8) $$b_{n+1}(x) = (2x - 1)b_n(x) = (2x - 1)^n.$$

PROPERTIES OF $a_i(x)$, $b_i(x)$

Notice that

(9) $$a_n(x) = b_n(x) - b_{n-1}(x) \qquad (n \geq 2)$$

and

(10) $$\frac{a_n(x)}{a_{n-1}(x)} = \frac{b_n(x)}{b_{n-1}(x)} = (2x - 1) \qquad (n > 2)$$

Write

(11) $$b \equiv b(x,t) = [1 - (2x - 1)t]^{-1} = \sum_{n=1}^{\infty} b_n(x)t^{n-1};$$

(12) $$a \equiv a(x,t) = (2x - 2)[1 - (2x - 1)t]^{-1} = \sum_{n=2}^{\infty} a_n(x)t^{n-2}.$$

Calculations yield

(13) $$2t\frac{\partial b}{\partial t} - (2x - 1)\frac{\partial b}{\partial x} = 0;$$

(14) $$2t\frac{\partial a}{\partial t} - (2x - 1)\frac{\partial a}{\partial x} + 2(2x - 1)b = 0.$$

Also

(15) $$(2x - 1)b_n'(x) - 2(n - 1)b_n(x) = 0,$$

(16) $$(2x - 1)a_{n+2}'(x) - 2(n + 1)a_{n+2}(x) - 2(2x - 1)b_n(x) = 0,$$

where the prime (dash) represents the first derivative w.r.t. x.

Results (9), (10), and (13)-(16) should be compared with corresponding results in [2] for the class of sequences studied there.

PART II

RISING AND DESCENDING DIAGONAL FUNCTIONS FOR FERMAT POLYNOMIALS

The First Fermat Polynomials $\phi_n(x)$; The Second Fermat Polynomials $\theta_n(x)$

The sequence $\{\phi_n\} = \{0, 1, 3, 7, 15, \ldots\}$ for which

(17′) $$\phi_{n+2} = 3\phi_{n+1} - 2\phi_n \qquad \phi_0 = 0,\ \phi_1 = 1 \qquad (n \geq 0)$$

is generalized to the *first Fermat polynomial sequence* $\{\phi_n(x)\}$ for which

(17) $$\phi_{n+2}(x) = x\phi_{n+1}(x) - 2\phi_n(x) \qquad \phi_0(x) = 0,\ \phi_1(x) = 1 \qquad (n \geq 0).$$

Similarly, the sequence $\{\theta_n\} = \{2, 3, 5, 9, \ldots\}$ for which

(18′) $$\theta_{n+2} = 3\theta_{n+1} - 2\theta_n \qquad \theta_0 = 2,\ \theta_1 = 3 \qquad (n \geq 0)$$

is generalized to the *second Fermat polynomial sequence* $\{\theta_n(x)\}$ for which

(18) $$\theta_{n+2}(x) = x\theta_{n+1}(x) - 2\theta_n(x) \qquad \theta_0(x) = 2,\ \theta_1(x) = x \qquad (n \geq 0).$$

Terms of these sequences are as follows:

(19)
$$\begin{cases} \phi_0(x) = 0 \\ \phi_1(x) = 1 \\ \phi_2(x) = x \\ \phi_3(x) = x^2 - 2 \\ \phi_4(x) = x^3 - 4x \\ \phi_5(x) = x^4 - 6x^2 + 4 \\ \phi_6(x) = x^5 - 8x^3 + 12x \\ \phi_7(x) = x^6 - 10x^4 + 24x^2 - 8 \\ \phi_8(x) = x^7 - 12x^5 + 40x^3 - 32x \\ \phi_9(x) = x^8 - 14x^6 + 60x^4 - 80x^2 + 16 \end{cases}$$

....

(20)
$$\begin{cases} \theta_0(x) = 2 \\ \theta_1(x) = x \\ \theta_2(x) = x^2 - 4 \\ \theta_3(x) = x^3 - 6x \\ \theta_4(x) = x^4 - 8x^2 + 8 \\ \theta_5(x) = x^5 - 10x^3 + 20x \\ \theta_6(x) = x^6 - 12x^4 + 36x^2 - 16 \\ \theta_7(x) = x^7 - 14x^5 + 56x^3 - 56x \\ \theta_8(x) = x^8 - 16x^6 + 80x^4 - 128x^2 + 32 \\ \theta_9(x) = x^9 - 18x^7 + 108x^5 - 240x^3 + 144x \end{cases}$$

....

RISING AND DESCENDING DIAGONAL FUNCTIONS FOR $\phi_n(x)$, $\theta_n(x)$

Label the rising and descending diagonal functions

$$R_i(x),\ D_i(x) \quad \text{for} \quad \{\phi_n(x)\}$$

and

$$R_i'(x),\ D_i'(x) \quad \text{for} \quad \{\theta_n(x)\}.$$

Of course, in this context the primes do *not* represent derivatives.

Reading from the listed information in (19) and (20),

$$\text{if } D_1(x) = 1,\ D_1'(x) = 2,$$

we have,

(21) $$D_n(x) = (x - 2)^{n-1},$$

(22) $$D_n'(x) = (x - 4)(x - 2)^{n-2} \qquad (n \geq 2)$$

whence

(23)
$$\begin{cases} \dfrac{D_{n+1}(x)}{D_n(x)} = \dfrac{D'_{n+1}(x)}{D'_n(x)} = x - 2 & (n \geq 2) \\ \dfrac{D'_n(x)}{D_n(x)} = \dfrac{x-4}{x-2} & (n \geq 2;\ x \neq 2) \\ \dfrac{D'_{n+1}(x)}{D_n(x)} = x - 4 & \end{cases}$$

Also

(24) $$D_n(x) - 2D_{n-1}(x) = D'_n(x).$$

Rising diagonal functions may be tabulated thus:

	i =	1	2	3	4	5	6	7	8	...
(25)	$R_i(x)$	1	x	x^2	x^3-2	x^4-4x	x^5-6x^2	x^6-8x^3+4	x^7-10x^4+12x	...
(26)	$R'_i(x)$	2	x	x^2	x^3-4	x^4-6x	x^5-8x^2	x^6-10x^3+8	x^7-12x^4+20x	...

with the properties ($n > 3$),

(27)
$$\begin{cases} R'_n(x) = R_n(x) - 2R_{n-3}(x) \\ R_n(x) = xR_{n-1}(x) - 2R_{n-3}(x) \\ R'_n(x) = xR'_{n-1}(x) - 2R'_{n-3}(x). \end{cases}$$

Calculations of results similar to those in (13)-(16) follow as a matter of course for both rising and descending diagonal functions, but these are left for the curious reader. (A comparison with corresponding results in [2] is desirable.)

However, it is worthwhile to record the generating functions for the diagonal functions associated with the two Fermat sequences. These are, for $D_i(x)$, $D'_i(x)$, $R_i(x)$, $R'_i(x)$, respectively:

(28) $$\sum_{n=1}^{\infty} D_n(x)t^{n-1} = [1 - (x-2)t]^{-1};$$

(29) $$\sum_{n=2}^{\infty} D'_n(x)t^{n-2} = (x-4)[1-(x-2)t]^{-1};$$

(30 $$\sum_{n=1}^{\infty} R_n(x)t^{n-1} = [1 - (xt - 2t^3)]^{-1};$$

(31) $$\sum_{n=2}^{\infty} R'_n(x)t^{n-1} = (1-2t^3)[1-(xt-2t^3)]^{-1}.$$

It is expected that the results of [1], [2], and [3] will be generalized in a subsequent paper.

REFERENCES

1. A. F. Horadam. "Polynomials Associated with Chebyshev Polynomials of the First Kind." *The Fibonacci Quarterly* 15, No. 3 (1977):255-257.
2. A. F. Horadam. "Diagonal Functions." *The Fibonacci Quarterly* 16, No. 1 (1978):33-36.
3. D. V. Jaiswal. "On Polynomials Related to Tchebichef Polynomials of the Second Kind." *The Fibonacci Quarterly* 12, No. 3 (1974):263-265.

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ON EULER'S SOLUTION OF A PROBLEM OF DIOPHANTUS

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1. The four numbers 1, 3, 8, 120 have the property that the product of any two of them is one less than a square. This fact was apparently discovered by Fermat. As one of the first applications of Baker's method in Diophantine approximations, Baker and Davenport [2] showed that there is no fifth positive integer n, so that

$$n + 1,\ 3n + 1,\ 8n + 1,\ \text{and}\ 120n + 1$$

are all squares. It is not known how large a set of positive integers $\{x_1, x_2, \ldots, x_n\}$ can be found so that all $x_ix_j + 1$ are squares for all $1 \le i < j \le n$.

A solution attributed to Euler [1] shows that for every triple of integers x_1, x_2, y for which $x_1x_2 + 1 = y^2$ it is possible to find two further integers x_3, x_4 expressed as polynomials in x_1, x_2, y and a rational number x_5, expressed as a rational function in x_1, x_2, y; so that $x_ix_j + 1$ is the square of a rational expression x_1, x_2, y for all $1 \le i < j \le 5$.

In this note we analyze Euler's solution from a more abstract algebraic point of view. That is, we start from a field k of characteristic $\neq 2$ and adjoin independent transcendentals $x_1, x_2, \ldots, x_m$. We then set $x_ix_j + 1 = y_{ij}^2$ and pose two problems:

I. Find nonzero elements $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n$ in the ring $R = k[x_1, \ldots, x_m; y_{12}, \ldots, y_{m-1,m}]$ so that $x_ix_j + 1 = y_{ij}^2$; and $y_{ij} \in R$ for $1 \le i < j \le n$.

II. Find nonzero elements $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n$ in the field $K = k(x_1, \ldots, x_m; y_{12}, \ldots, y_{m-1,m})$ so that $x_ix_j + 1 = y_{ij}^2$; and $y_{ij} \in K$ for all $1 \le i < j \le n$.

In Section 2 we give a complete solution to Problem I for $m = 2$, $n = 3$. In Section 3 we give solutions for $m = 2$, $n = 4$ which include both Euler's

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