AN ALGORITHM FOR PACKING COMPLEMENTS OF FINITE SETS OF INTEGERS

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ABSTRACT

Let $A_k = \{0 = a_1 < a_2 < \dots < a_k\}$ and $B = \{0 = b_1 < b_2 < \dots < b_n \dots\}$ be sets of k integers and infinitely many integers, respectively. Suppose B has asymptotic density $x : d(B) = x$. If, for every integer $n \geq 0$, there is at most one representation $n = a_i + b_j$, then we say that A_k has a packing complement of density $\geq x$.

Given A_k and x, there is no known algorithm for determining whether or not B exists.

We define "regular packing complement" and give an algorithm for determining if B exists when packing complement is replaced by regular packing complement. We exemplify with the case $k = 5$, i.e., given A_5 and $x = 1/10$, we give an algorithm for determining if A_5 has a regular packing complement B with density $\geq 1/10$. We relate this result to the

Conjecture: Every A_5 has a packing complement of density $\geq 1/10$.

Let

$$A_k = \{0 = a_1 < a_2 < \dots < a_k\}$$

and

$$B = \{0 = b_1 < b_2 < \dots < b_n < \dots\}$$

be sets of k integers and infinitely many integers, respectively. If, for every integer $n \geq 0$, $n = a_i + b_j$ has at most one solution, then we call B a packing complement, or p-complement, of A_k.

Let $B(n)$ denote the counting function of B and define $d(B)$, the density of B, as follows:

$$d(B) = \lim_{n \to \infty} B(n)/n \quad \text{if this limit exists.}$$

From now on we consider only those sets B for which the density exists.

For a given set A_k, we wish to find the p-complement B with maximum density. More precisely, we define $p(A_k)$, the packing codensity of A_k, as follows:

$$p(A_k) = \sup_B d(B) \quad \text{where } B \text{ ranges over all } p\text{-complements of } A_k.$$

Finally, we define p_k as the "smallest" p-codensity of any A_k, or, more precisely,

$$p_k = \inf_{A_k} p(A_k).$$

We proved [1] that, for $\varepsilon > 0$,

$$\frac{1}{\binom{k}{2} + 1} \leq p_k \leq \frac{2.66\dots}{k^2} + \varepsilon$$

if k is sufficiently large.

The first four p_k are trivial, since we can find sets for which the lower bound is attained. Thus,

$$A_1 = \{0\},\ A_2 = \{0,1\},\ A_3 = \{0,\ 1,\ 3\},\ A_4 = \{0,\ 1,\ 4,\ 6\}$$

give

$$p_1 = 1,\ p_2 = 1/2,\ p_3 = 1/4,\ p_4 = 1/7.$$

But,

$$\frac{1}{11} \le p_5 \le \frac{1}{10}.$$

The upper bound is established by $A_5 = \{0, 1, 2, 6, 9\}$ and the lack of certainty in the lower bound is caused by the impossibility of finding A_5 whose difference set takes on all values 1, 2, ..., 10.

Suppose we have a set A_k, a set $B = \{b_1, b_2, \ldots, b_n\}$, and a number N such that $a + b \equiv m \pmod N$ has at most one solution,

$$a \in A_k,\ b \in B, \quad \text{for } 0 \le m < N.$$

Then the packing codensity of A_k is $\ge n/N$.

If, in the previous paragraph, the p-complement B consists entirely of consecutive multiples of M, where $(M,N) = 1$, i.e., $B = \{M, 2M, \ldots, nM\} \pmod N$, then we say that A_k has a *regular p-complement* of density $\ge n/N$.

As in [2], there is no known algorithm for determining either the packing codensity of A_k or even whether A_k has a p-complement of density $\ge x$.

It is the purpose of this note to give an algorithm for answering the question: does A_k have a regular p-complement of density $\ge x$? We actually give a method for determining whether A_5 has a regular p-complement of density $\ge 1/10$, because of its application to the

Conjecture: $p_5 = 1/10$.

However, the generalization of our result is obvious.

We adopt the following conventions throughout:

(1) A_5 represents a set of five integers,

$$A_5 = \{0 = a_1 < a_2 < a_3 < a_4 < a_5\}.$$

(2) M and N are positive integers, with $M < N$, $(M,N) = 1$.

(3) All a_i are distinct mod N.

(4) "a_i and a_j are adjacent mod N" means that for some M the residues mod N of a_i and a_j occur in the ordered N-tuple $\{M, 2M, \ldots, NM\} \pmod N$ with residue mod N of no other element a_k between them. We illustrate with

$$A_5 = \{0, 1, 24, 25, 28\},\ N = 13,\ M = 5.$$

The ordered 13-tuple is

$$\{5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8, 0\}$$

and since

$$\{0, 1, 24, 25, 28\} \equiv \{0, 1, 2, 11, 12\} \pmod{13},$$

we can write

$$A_5 \equiv \{0, 1, 2, 11, 12\} \pmod{13}.$$

In the ordered 13-tuple, A_5 has the following adjacent pairs:

$$\{0, 11\},\ \{11, 1\},\ \{1, 12\},\ \{12, 2\},\ \{2, 0\}.$$

But $\{11, 12\}$ are not adjacent, because 1 is between them in one sense and 0 and 2 are between them in the opposite sense. Similarly,

$$\{1, 2\},\ \{0, 1\},\ \{2, 11\}, \text{ and } \{0, 12\}$$

are nonadjacent pairs.

(5) "A_5 has a regular p-complement" will mean that it has a regular p-complement of density $\geq 1/10$.

Lemma 1: Given A_5, let a_i and a_j be adjacent mod N and write

$$d_{ij}M \equiv a_i - a_j \pmod{N}.$$

Then A_5 has a regular p-complement if and only if

$$\frac{N}{10} \leq d_{ij},\ d_{ji} < N,$$

for all five adjacent pairs i,j.

Proof: Let $C = \{M, 2M, \ldots, NM\}$ (mod N) be an ordered N-tuple. Since $a_1, \ldots, a_5$ will occur in C in some order as distinct residues mod N, we assume, without loss of generality, that $0 \leq a_i < N$, $i = 1, \ldots, 5$. Assume that a_j is to the left of a_i in C. (Zero is to the left of the first a_k in C.) Write

$$B = \left\{M,\ 2M,\ \ldots,\ \frac{N}{10}M\right\} \pmod{N}.$$

Suppose now that $N/10 < d_{ij},\ d_{ji} < N$. Then $a_j \oplus B$ includes the $N/10$ elements of C immediately to the right of a_j. Thus, while it may include a_i, it will not include any element to the right of a_i nor, of course, will it include a_j. Hence, $A_5 \oplus B$ cannot include any element of C more than once. Since C is a complete residue system mod N, B is a p-complement of A_5. Conversely, if $0 < d_{ij} < N/10$ or $0 < d_{ji} < N/10$, then

$$(a_j \oplus B) \cap (a_i \oplus B) \neq \phi$$

and B is not a p-complement of B.

Lemma 2: Given A_5, consider the congruence

(1) $$d_{ij}M \equiv a_i - a_j \pmod{N}.$$

Then A_5 has a regular p-complement if and only if there exists a solution of (1), with $N/10 \leq d_{ij} \leq 9N/10$, for every pair i,j, with $1 \leq i,\ j \leq 5$, $i \neq j$.

Proof: If A_5 has a regular p-complement, then Lemma 1 implies that

$$\frac{N}{10} \leq d_{ij},\ d_{ji} < N \quad \text{if } a_i \text{ and } a_j \text{ are adjacent mod } N.$$

This, in turn, implies that

$$\frac{N}{10} \leq d_{ij},\ d_{ji} \leq \frac{9N}{10}.$$

Clearly, the inequalities still hold if a_i and a_j are not adjacent mod N. If (1) has the required solution for every pair i,j, this implies that adjacent a's, mod N, are separated by at least $(N/10)M$, and so, by Lemma 1, A_5 has a regular p-complement.

Define k_0 by $k_0M \equiv 1 \pmod{N}$ and write $r = k_0/N$. Let $D_{ij} = a_i - a_j$. We have

Lemma 3: The congruence

(2) $$d_{ij}M \equiv a_i - a_j \pmod{N}$$

has a solution $N/10 \leq d_{ij} \leq 9N/10$ if and only if r satisfies one of the inequalities:

$$\frac{10(k-1)+1}{10|D_{ij}|} \le r \le \frac{10(k-1)+9}{10|D_{ij}|}, \quad k = 1, 2, \ldots, |D_{ij}|.$$

Proof: Suppose $\frac{N}{10} \le d_{ij} \le \frac{9N}{10}$. We have $d_{ij}M \equiv D_{ij} \pmod{N}$. However, since $k_0M \equiv 1 \pmod{N}$, we also have

$$D_{ij}k_0M \equiv D_{ij} \pmod{N}, \text{ so that}$$
$$d_{ij} \equiv D_{ij}k_0 \pmod{N}.$$

Therefore, $D_{ij}r \equiv s \pmod{1}$ where $\frac{1}{10} \le s \le \frac{9}{10}$.

This implies that

$$\frac{10(k-1)+1}{10} \le |D_{ij}|r \le \frac{10(k-1)+9}{10}$$

or

$$\frac{10(k-1)+1}{10|D_{ij}|} \le r \le \frac{10(k-1)+9}{10|D_{ij}|} \quad \text{for some } k,\ 1 \le k \le |D_{ij}|.$$

The argument can also be read backwards, so this completes the proof.

Since each difference D_{ij} determines a set of intervals R_{ij} on the unit interval:

$$R_{ij} = \bigcup_{k=1}^{|D_{ij}|} \left[\frac{10(k-1)+1}{10|D_{ij}|}, \frac{10(k-1)+9}{10|D_{ij}|}\right],$$

our result can be expressed in the following

Theorem: A_5 does not have a regular p-complement if and only if

(3) $$\bigcap_{1 \le i < j \le 5} R_{ij} = \phi$$

Proof: From Lemma 3 we see that every solution, $r = k_0/N$, to the congruence

$$d_{ij}M \equiv a_i - a_j \pmod{N}, \quad \frac{N}{10} \le d_{ij} \le \frac{9N}{10}$$

must lie in R_{ij}. By Lemma 2 we see that for A_5 to have a regular p-complement it is necessary and sufficient that this congruence have a simultaneous solution for every pair $1 \le i,\ j \le 5$. Hence,

$$\bigcap_{1 \le i < j \le 5} R_{ij} \ne \phi$$

if and only if A_5 has a regular p-complement.

The application of this theorem to a given A_5 is a tedious procedure without a computer. In [2], we stated that a computer search revealed two sets A_4, with $a_4 \le 100$, that do not have regular (covering) complements of density $\le 1/3$. We have no such computer information on the packing algorithm but still think it likely that at most a finite number of A_5's do not have regular p-complements. The obvious attempt to prove this is to assume a_5 is large and that (3) is satisfied. So far, we have failed to find the desired contradiction.

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ADDENDA TO "PYTHAGOREAN TRIPLES CONTAINING FIBONACCI NUMBERS: SOLUTIONS FOR $F_n^2 \pm F_k^2 = K^2$"

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In a recent correspondence from J. H. E. Cohn, it was learned that Ljunggren [1] has proved that the only square Pell numbers are 0, 1, and 169. (This appears as an unsolved problem, H-146, in [2] and as Conjecture 2.3 in [3].) Also, if the Fibonacci polynomials $\{F_n(x)\}$ are defined by

$$F_0(x) = 0,\ F_1(x) = 1, \text{ and } F_{n+2}(x) = xF_{n+1}(x) + F_n(x),$$

then the Fibonacci numbers are given by $F_n = F_n(1)$, and the Pell numbers are $P_n = F_n(2)$. Cohn [4] has proved that the only perfect squares among the sequences $\{F_n(a)\}$, a odd, are 0 and 1, and whenever $a = k^2$, a itself. Certain cases are known for a even [5].

The cited results of Cohn and Ljunggren mean that Conjectures 2.3, 3.2, and 4.2 of [3] are true, and that the earlier results can be strengthened as follows.

If $(n,k) = 1$, there are no solutions in positive integers for

$$F_n^2(a) + F_k^2(a) = K^2,\ n > k > 0, \text{ when } a \text{ is odd and } a \geq 3.$$

This is the same as stating that no two members of $\{F_n(a)\}$ can occur as the lengths of legs in a primitive Pythagorean triangle, for a odd and $a \geq 3$.

When $a = 1$, for Fibonacci numbers, if

$$F_n^2 + F_k^2 = K^2,\quad n > k > 0,$$

then $(n,k) = 2$, and it is conjectured that there is no solution in positive integers. When $a = 2$, for Pell numbers, $P_n^2 + P_k^2 = K^2$ has the unique solution $n = 4$, $k = 3$, giving the primitive Pythagorean triple 5—12—13.

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