AN ALGORITHM FOR PACKING COMPLEMENTS OF
FINITE SETS OF INTEGERS

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ABSTRACT

Let \( A_k = \{0 = a_1 < a_2 < \ldots < a_k \} \) and \( B = \{0 = b_1 < b_2 < \ldots < b_n \ldots \} \) be sets of \( k \) integers and infinitely many integers, respectively. Suppose \( B \) has asymptotic density \( \bar{x} : \bar{d}(B) = \bar{x} \). If, for every integer \( n \geq 0 \), there is at most one representation \( n = a_l + b_j \), then we say that \( A_k \) has a packing complement of density \( \bar{x} \).

Given \( A_k \) and \( \bar{x} \), there is no known algorithm for determining whether or not \( B \) exists.

We define "regular packing complement" and give an algorithm for determining if \( B \) exists when packing complement is replaced by regular packing complement. We exemplify with the case \( k = 5 \), i.e., given \( A_5 \) and \( \bar{x} = 1/10 \), we give an algorithm for determining if \( A_5 \) has a regular packing complement \( B \) with density \( \geq 1/10 \).

We relate this result to the

**Conjecture:** Every \( A_5 \) has a packing complement of density \( \geq 1/10 \). Let

\[
A_k = \{0 = a_1 < a_2 < \ldots < a_k\}
\]

and

\[
B = \{0 = b_1 < b_2 < \ldots < b_n < \ldots\}
\]

be sets of \( k \) integers and infinitely many integers, respectively. If, for every integer \( n \geq 0 \), \( n = a_l + b_j \) has at most one solution, then we call \( B \) a packing complement, or \( p \)-complement, of \( A_k \).

Let \( B(n) \) denote the counting function of \( B \) and define \( d(B) \), the density of \( B \), as follows:

\[
d(B) = \lim_{n \to \infty} B(n)/n \text{ if this limit exists.}
\]

From now on we consider only those sets \( B \) for which the density exists.

For a given set \( A_k \), we wish to find the \( p \)-complement \( B \) with maximum density. More precisely, we define \( p(A_k) \), the packing codensity of \( A_k \), as follows:

\[
p(A_k) = \sup_B d(B) \text{ where } B \text{ ranges over all } p \text{-complements of } A_k.
\]

Finally, we define \( p_k \) as the "smallest" \( p \)-codensity of any \( A_k \), or, more precisely,

\[
p_k = \inf_{A_k} p(A_k).
\]

We proved [1] that, for \( \varepsilon > 0 \),

\[
\frac{1}{\binom{k}{2} + 1} \leq p_k \leq \frac{2.66\ldots + \varepsilon}{k^2}
\]

if \( k \) is sufficiently large.

The first four \( p_k \) are trivial, since we can find sets for which the lower bound is attained. Thus,

\[
A_1 = \{0\}, A_2 = \{0,1\}, A_3 = \{0,1,3\}, A_4 = \{0,1,4,6\}
\]
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give

\[ p_1 = 1, p_2 = 1/2, p_3 = 1/4, p_4 = 1/7. \]

But,

\[ \frac{1}{11} \leq p_5 \leq \frac{1}{10}. \]

The upper bound is established by \( A_5 = \{0, 1, 2, 6, 9\} \) and the lack of certainty in the lower bound is caused by the impossibility of finding \( A_5 \) whose difference set takes on all values 1, 2, \ldots, 10.

Suppose we have a set \( A_k \), a set \( B = \{b_1, b_2, \ldots, b_n\} \), and a number \( N \) such that \( \alpha + \beta \equiv m \pmod{N} \) has at most one solution,

\[ \alpha \in A_k, \beta \in B, \text{ for } 0 \leq m < N. \]

Then the packing codensity of \( A_k \) is \( \geq n/N \).

If, in the previous paragraph, the \( p \)-complement \( B \) consists entirely of consecutive multiples of \( M \), where \( (M,N) = 1 \), i.e., \( B = \{M, 2M, \ldots, nM\} \pmod{N} \), then we say that \( A_k \) has a regular \( p \)-complement of density \( \geq n/N \).

As in [2], there is no known algorithm for determining either the packing codensity of \( A_k \) or even whether \( A_k \) has a \( p \)-complement of density \( \geq \pi \).

It is the purpose of this note to give an algorithm for answering the question: does \( A_k \) have a regular \( p \)-complement of density \( \geq \pi \)? We actually give a method for determining whether \( A_5 \) has a regular \( p \)-complement of density \( \geq 1/10 \), because of its application to the

**Conjecture:** \( p_5 = 1/10. \)

However, the generalization of our result is obvious.

We adopt the following conventions throughout:

1. \( A_5 \) represents a set of five integers,
   \[ A_5 = \{0 = \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5\}. \]

2. \( M \) and \( N \) are positive integers, with \( M < N \), \( (M,N) = 1 \).

3. All \( \alpha_i \) are distinct \( \pmod{N} \).

4. "\( \alpha_i \) and \( \alpha_j \) are adjacent \( \pmod{N} \)" means that for some \( M \) the residues \( \pmod{N} \) of \( \alpha_i \) and \( \alpha_j \) occur in the ordered \( N \)-tuple \( \{M, 2M, \ldots, nM\} \pmod{N} \) with residue \( \pmod{N} \) of no other element \( \alpha_k \) between them. We illustrate with
   \[ A_5 = \{0, 1, 24, 25, 28\}, \quad N = 13, \quad M = 5. \]

The ordered 13-tuple is
\[ \{5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8, 0\} \]

and since
\[ \{0, 1, 24, 25, 28\} \equiv \{0, 1, 2, 11, 12\} \pmod{13}, \]

we can write
\[ A_5 \equiv \{0, 1, 2, 11, 12\} \pmod{13}. \]

In the ordered 13-tuple, \( A_5 \) has the following adjacent pairs:
\[ \{0, 11\}, \{11, 1\}, \{1, 12\}, \{12, 2\}, \{2, 0\}. \]

But \( \{11, 12\} \) are not adjacent, because 1 is between them in one sense and 0 and 2 are between them in the opposite sense. Similarly,
\[ \{1, 2\}, \{0, 1\}, \{2, 11\}, \text{ and } \{0, 12\} \]

are nonadjacent pairs.
(5) "$A_5$ has a regular $p$-complement" will mean that it has a regular $p$-
complement of density $\geq 1/10$.

**Lemma 1**: Given $A_5$, let $a_i$ and $a_j$ be adjacent mod $N$ and write
\[ d_{ij}M \equiv a_i - a_j \pmod{N}. \]
Then $A_5$ has a regular $p$-complement if and only if
\[ \frac{N}{10} \leq d_{ij}, d_{ji} < N, \]
for all five adjacent pairs $i,j$.

**Proof**: Let $C = \{M, 2M, \ldots, 10M\} \pmod{N}$ be an ordered $N$-tuple. Since $a_1,
\ldots, a_5$ will occur in $C$ in some order as distinct residues mod $N$, we assume,
without loss of generality, that $0 \leq a_i < N$, $i = 1, \ldots, 5$. Assume that $a_j$
is to the left of $a_i$ in $C$. (Zero is to the left of the first $a_k$ in $C$.) Write
\[ B = \left\{ \frac{M}{10}, \frac{2M}{10}, \ldots, \frac{N}{10} \right\} \pmod{N}. \]
Suppose now that $N/10 < d_{ij}, d_{ji} < N$. Then $a_j \oplus B$ includes the $N/10$
elements of $C$ immediately to the right of $a_j$. Thus, while it may include $a_i$, it will
not include any element to the right of $a_i$ nor, of course, will it include
$a_j$. Hence, $A_5 \oplus B$ cannot include any element of $C$ more than once. Since $C$
is a complete residue system mod $N$, $B$ is a $p$-complement of $A_5$. Conversely,
if $0 < d_{ij} < N/10$ or $0 < d_{ji} < N/10$, then
\[ (a_j \oplus B) \cap (a_i \oplus B) \neq \emptyset \]
and $B$ is not a $p$-complement of $B$.

**Lemma 2**: Given $A_5$, consider the congruence
\[ d_{ij}M \equiv a_i - a_j \pmod{N}. \]
Then $A_5$ has a regular $p$-complement if and only if there exists a solution of
(1), with $N/10 < d_{ij}, d_{ji} < 9N/10$, for every pair $i,j$, with $1 \leq i, j \leq 5, i \neq j$.

**Proof**: If $A_5$ has a regular $p$-complement, then Lemma 1 implies that
\[ \frac{N}{10} \leq d_{ij}, d_{ji} < N \text{ if } a_i \text{ and } a_j \text{ are adjacent mod } N. \]
This, in turn, implies that
\[ \frac{N}{10} \leq d_{ij}, d_{ji} \leq \frac{9N}{10}. \]
Clearly, the inequalities still hold if $a_i$ and $a_j$ are not adjacent mod $N$. If
(1) has the required solution for every pair $i,j$, this implies that adjacent
$a$'s, mod $N$, are separated by at least $(N/10)M$, and so, by Lemma 1, $A_5$
has a regular $p$-complement.

Define $k_0$ by $k_0M \equiv 1 \pmod{N}$ and write $r = k_0/N$. Let $D_{ij} = a_i - a_j$. We have

**Lemma 3**: The congruence
\[ d_{ij}M \equiv a_i - a_j \pmod{N} \]
has a solution $N/10 \leq d_{ij} \leq 9N/10$ if and only if $r$ satisfies one of the in-
equalities:
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\[
\frac{10(k - 1) + 1}{10|D_{\xi \eta}|} \leq r \leq \frac{10(k - 1) + 9}{10|D_{\xi \eta}|}, \quad k = 1, 2, \ldots, |D_{\xi \eta}|.
\]

**Proof:** Suppose \(\frac{N}{10} \leq d_{\xi \eta} \leq \frac{9N}{10}\). We have \(d_{\xi \eta}M \equiv D_{\xi \eta} \pmod{N}\). However, since \(k_0M \equiv 1 \pmod{N}\), we also have

\[
D_{\xi \eta}k_0M \equiv D_{\xi \eta} \pmod{N},
\]
so that

\[
d_{\xi \eta} \equiv D_{\xi \eta}k_0 \pmod{N}.
\]

Therefore, \(D_{\xi \eta}r \equiv s \pmod{1}\) where \(\frac{1}{10} \leq s \leq \frac{9}{10}\).

This implies that

\[
\frac{10(k - 1) + 1}{10} \leq |D_{\xi \eta}|r \leq \frac{10(k - 1) + 9}{10}
\]
or

\[
\frac{10(k - 1) + 1}{10|D_{\xi \eta}|} \leq r \leq \frac{10(k - 1) + 9}{10|D_{\xi \eta}|}
\]
for some \(k, 1 \leq k \leq |D_{\xi \eta}|\).

The argument can also be read backwards, so this completes the proof.

Since each difference \(D_{\xi \eta} \) determines a set of intervals \(R_{\xi \eta} \) on the unit interval:

\[
R_{\xi \eta} = \bigcup_{k=1}^{[a_{\xi \eta}]} \left[ \frac{10(k - 1) + 1}{10|D_{\xi \eta}|}, \frac{10(k - 1) + 9}{10|D_{\xi \eta}|} \right]
\]
our result can be expressed in the following

**Theorem:** \(A_5\) does not have a regular \(p\)-complement if and only if

\[
\bigcap_{1 \leq i < j \leq 5} R_{\xi \eta} = \phi
\]

**Proof:** From Lemma 3 we see that every solution, \(r = k_0/N\), to the congruence

\[
d_{\xi \eta}M \equiv a_{\xi} - a_{\eta} \pmod{N}, \quad \frac{N}{10} \leq d_{\xi \eta} \leq \frac{9N}{10}
\]
must lie in \(R_{\xi \eta}\). By Lemma 2 we see that for \(A_5\) to have a regular \(p\)-complement it is necessary and sufficient that this congruence have a simultaneous solution for every pair \(1 \leq i, j \leq 5\). Hence,

\[
\bigcap_{1 \leq i < j \leq 5} R_{\xi \eta} \neq \phi
\]
if and only if \(A_5\) has a regular \(p\)-complement.

The application of this theorem to a given \(A_5\) is a tedious procedure without a computer. In [2], we stated that a computer search revealed two sets \(A_5\), with \(a_5 \leq 100\), that do not have regular (covering) complements of density \(\leq 1/3\). We have no such computer information on the packing algorithm but still think it likely that at most a finite number of \(A_5\)'s do not have regular \(p\)-complements. The obvious attempt to prove this is to assume \(a_5\) is large and that (3) is satisfied. So far, we have failed to find the desired contradiction.
REFERENCES


ADDENDA TO "PYTHAGOREAN TRIPLES CONTAINING FIBONacci NUMBERS": SOLUTIONS FOR $F_n^2 + F_k^2 = K^2$

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In a recent correspondence from J. H. E. Cohn, it was learned that Ljunggren [1] has proved that the only square Pell numbers are 0, 1, and 169. (This appears as an unsolved problem, H-146, in [2] and as Conjecture 2.3 in [3].) Also, if the Fibonacci polynomials $\{F_n(x)\}$ are defined by

$$F_0(x) = 0, F_1(x) = 1, \text{ and } F_{n+2}(x) = x F_{n+1}(x) + F_n(x),$$

then the Fibonacci numbers are given by $F_n = F_n(1)$, and the Pell numbers are $P_n = F_n(2)$. Cohn [4] has proved that the only perfect squares among the sequences $\{F_n(a)\}$, $a$ odd, are 0 and 1, and whenever $a = k^2$, $a$ itself. Certain cases are known for $a$ even [5].

The cited results of Cohn and Ljunggren mean that Conjectures 2.3, 3.2, and 4.2 of [3] are true, and that the earlier results can be strengthened as follows.

If $(n,k) = 1$, there are no solutions in positive integers for

$$F_n^2(a) + F_k^2(a) = K^2, \text{ } n > k > 0, \text{ when } a \text{ is odd and } a \geq 3.$$

This is the same as stating that no two members of $\{F_n(a)\}$ can occur as the lengths of legs in a primitive Pythagorean triangle, for $a$ odd and $a \geq 3$.

When $a = 1$, for Fibonacci numbers, if

$$F_n^2 + F_k^2 = K^2, \text{ } n > k > 0,$$

then $(n,k) = 2$, and it is conjectured that there is no solution in positive integers. When $a = 2$, for Pell numbers, $P_n^2 + P_k^2 = K^2$ has the unique solution $n = 4, k = 3$, giving the primitive Pythagorean triple 5-12-13.

REFERENCES


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