# SOME EXTENSIONS OF WYTHOFF PAIR SEQUENCES

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In [1] it is shown that if $\alpha = \frac{1+\sqrt{5}}{2}$ then

$$[[n\alpha]\alpha] = [n\alpha] + n - 1 \tag{1}$$

for all positive integers $n$. Out first purpose in this paper is to give an alternate proof of (1) and also show that (1) holds even if $n$ is negative. Next, we prove that the converse of (1) holds even if (1) is true for all negative integers. In conclusion, we derive an additional identity using the greatest integer function together with the golden ratio, and we discuss two sets of sequences related to these results.

First we show

*Theorem 1:* If $\delta = \frac{1+\sqrt{5}}{2}$ then $[[n\delta]\ ] = [n\delta] + n - 1$ for all integers $n \neq 0$.

Before proving Theorem 1, let us recall a theorem of Skolem and Bang which can be found in [2].

*Theorem 2:* Let $\varepsilon$ and $t$ be positive real numbers. Denote the set of all positive integers by $Z$ and the null set by $\emptyset$. Let $N_\gamma = \{[n\gamma]\}_{n=1}^{\infty}$. Then $N_\varepsilon \cap N_t = \emptyset$ and $N_\varepsilon \cup N_t = Z$ if and only if $\varepsilon$ and $t$ are irrational and $\varepsilon^{-1} + t^{-1} = 1$.

*Proof of Theorem 1:* Let us assume that $n$ is positive. Since $n\delta$ is not an integer for any $n \neq 0$, we have $[n\delta] < n\delta < [n\delta] + 1$ provided $n \neq 0$, so that

$$[n\delta]\delta < n\delta^2 < ([n\delta] + 1)\delta. \tag{2}$$

In Theorem 2, let $\varepsilon = \delta$ and $t = \delta^2$, then $\varepsilon^{-1} + t^{-1} = 1$, so that $N_\delta \cap N_{\delta^2} = \emptyset$ and $N_\delta \cup N_{\delta^2} = Z$. Because $[[n\delta]\delta]$ and $[([n\delta]+1)\delta]$ are elements of $N_\delta$, while $[n\delta^2]$ belongs to $N_{\delta^2}$, we know from (2) that

$$[[n\delta]\delta] < [n\delta^2] < [([n\delta] + 1)\delta]. \tag{3}$$

Using the well-known fact that $[a + b] = [a] + [b] + \gamma$ where $\gamma = 0$ or $1$, we see that $[[n\delta]\delta + \delta] = [[n\delta]\delta] + [\delta] + \gamma = [[n\delta]\delta] + 1 + \gamma$ where $\gamma = 0$ or 1. Since $[n\delta^2] - [[n\delta]\delta]$ is an integer, we conclude from (3) that

$$[n\delta^2] - [[n\delta]\delta] = 1 \tag{4}$$

and $\gamma = 1$. Recalling that $\delta^2 = \delta + 1$, we obtain

$$[[n\delta]\delta] = [n\delta] + n - 1 \tag{5}$$

and the theorem is proved if $n > 0$.

Let us now assume that $n < 0$ and recall that since $n\delta$ is not an integer then $[n\delta] = -[-n\delta] - 1$. Using this fact together with the results above for $n > 0$, we have

$$\begin{aligned}[[n\delta]\delta] &= -[-[n\delta]\delta] - 1 \\ &= -[([-n\delta] + 1)\delta] - 1\end{aligned}$$

$$\begin{aligned} &= -([[-n\delta]\delta] + [\delta]) - 2 \\ &= -([-n\delta] - n) - 2 \\ &= -(-[n\delta] - n) - 1 \\ &= [n\delta] + n - 1 \end{aligned}$$

and the theorem is proved.

We now show

*Theorem 3*: If $[[n\delta]\delta] = [n\delta] + n - 1$ for all integers $n \neq 0$, then $\delta = \frac{1+\sqrt{5}}{2}$.

*Proof*: Since $[[n\delta]\delta] = [n\delta] + n - 1$, we have $[n\delta] + n - 1 \leq [n\delta]\delta < [n\delta] + n$. Therefore, $1 < \frac{[n\delta](\delta - 1)}{n} \leq 1 - \frac{1}{n}$ when $n < 0$, while $1 - \frac{1}{n} \leq \frac{[n\delta](\delta - 1)}{n} < 1$ if $n > 0$.

Hence,

$$\text{(6)} \qquad \lim_{n \to 0} \frac{[n\delta]}{n} = \frac{1}{\delta - 1},$$

provided $\delta \neq 1$, which is obviously true.

By definition of the greatest integer, we know that $[n\delta] \leq n\delta < [n\delta] + 1$ for any integer $n$ and any $\delta$ so that

$$\delta - \frac{1}{n} < \frac{[n\delta]}{n} \leq \delta \text{ if } n > 0, \text{ while } \delta \leq \frac{[n\delta]}{n} < \delta - \frac{1}{n} \text{ when } n < 0.$$

In both cases,

$$\text{(7)} \qquad \lim_{n \to 0} \frac{[n\delta]}{n} = \delta.$$

Equating (6) and (7), we see that $\delta^2 - \delta - 1 = 0$. If $\delta = \frac{1-\sqrt{5}}{2}$ and $n = 1$ then $[[n\delta]\delta] = [-\delta] = 0$, while $[n\delta] + n - 1 = -1$. Hence (1) is false. Therefore, $\delta = \frac{1+\sqrt{5}}{2}$ and we are done.

Another identity which arose while investigating (1) is

*Theorem 4*: If $\delta = \frac{1+\sqrt{5}}{2}$, then $[[n\delta]\delta + n\delta] = 2[n\delta] + n$ for all integers $n$ and conversely.

The proof of Theorem 4 is omitted, since it is essentially the same as the proof of Theorems 1 and 3, with the only difficulty arising when trying to prove the result for $n < 0$. This difficulty is overcome by using the fact that

$$[[n\delta]\delta + n\delta + \delta] = [[n\delta]\delta + n\delta] + 1 \text{ when } n > 0 \text{ and } \delta = \frac{1+\sqrt{5}}{2}.$$

The argument for the validity of the last statement can be found in [3].

Let us now illustrate some interesting applications of Theorems 1, 3, and 4. To do so we introduce two special sets of sequences. For any integer $n \neq 0$, define $\{S_m(n)\}_{m=1}^{\infty}$ by

$$\text{(8)} \quad S_m(n) = S_{m-1}(n) + S_{m-2}(n),\ m \geq 3,\ S_1(n) = n,\ S_2(n) = [n\alpha],\ \alpha = \frac{1+\sqrt{5}}{2}$$

and for any integer $n$, define $\{S_m^*(n)\}_{m=1}^{\infty}$ by

(9) $$S_m^*(n) = S_{m-1}^*(n) + S_{m-2}^*(n) - 1,\ m \geq 3,$$
$$S_1^*(n) = n,\ S_2^*(n) = [n\alpha],\ \alpha = \frac{1+\sqrt{5}}{2}.$$

Since (8) is a generalized Fibonacci sequence, it is easy to show that

(10) $$S_m(n) = F_{m-1}[n\alpha] + nF_{m-2},\ n \neq 0 \text{ and } m \geq 1.$$

The terms of $\{S_m(n)\}_{m=1}^{\infty}$ for $1 \leq m \leq 7$ and $-10 \leq n \leq 10$, $n \neq 0$, are presented in Table 1.

*TABLE 1*

| $S_m(n)$ \ $n$ | -10 | -9 | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_1(n)$ | -10 | -9 | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 |
| $S_2(n)$ | -17 | -15 | -13 | -12 | -10 | -9 | -7 | -5 | -4 | -2 |
| $S_3(n)$ | -27 | -24 | -21 | -19 | -16 | -14 | -11 | -8 | -6 | -3 |
| $S_4(n)$ | -44 | -39 | -34 | -31 | -26 | -23 | -18 | -13 | -10 | -5 |
| $S_5(n)$ | -71 | -63 | -55 | -50 | -42 | -37 | -29 | -21 | -16 | -8 |
| $S_6(n)$ | -115 | -102 | -89 | -81 | -68 | -60 | -47 | -34 | -26 | -13 |
| $S_7(n)$ | -186 | -165 | -144 | -131 | -110 | -97 | -76 | -55 | -42 | -21 |
| $S_m(n)$ \ $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $S_1(n)$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $S_2(n)$ | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 12 | 14 | 16 |
| $S_3(n)$ | 2 | 5 | 7 | 10 | 13 | 15 | 18 | 20 | 23 | 26 |
| $S_4(n)$ | 3 | 8 | 11 | 16 | 21 | 24 | 29 | 32 | 37 | 42 |
| $S_5(n)$ | 5 | 13 | 18 | 26 | 34 | 39 | 47 | 52 | 60 | 68 |
| $S_6(n)$ | 8 | 21 | 29 | 42 | 55 | 63 | 76 | 84 | 97 | 110 |
| $S_7(n)$ | 13 | 31 | 47 | 68 | 89 | 102 | 123 | 136 | 157 | 178 |

One of the first observations made was that some of the rows appear to be subsets of previous rows. A more careful examination implies that for a specific $n$ the positive and negative values for a given row are related by the Fibonacci numbers. The latter result is stated as Theorem 5, while the former is Theorem 6.

*Theorem 5:* For all integers $n \neq 0$, $S_m(n) + S_m(-n) = -F_{m-1}$, $m \geq 1$.

The proof of Theorem 5 is a direct result of (10) and is thus omitted.

*Theorem 6:* For any integer $m \geq 3$ and any integer $n \neq 0$,

$$S_m(n) = S_{m-2}(S_3(n)).$$

*Proof*: By definition, $S_1(S_3(n)) = S_3(n)$, so the theorem is true if $m = 3$. By Theorem 4 we have

$$S_4(n) = S_3(n) + S_2(n) = 2[n\alpha] + n = [[n\alpha]\alpha + n\alpha]$$
$$= S_2([n\alpha] + n) = S_2(S_3(n)),$$

so that the result is true for $m = 4$.

Assuming the theorem true for all positive integers $m \le k$ where $k \ge 4$, we have

$$S_{k+1}(n) = S_k(n) + S_{k-1}(n) = S_{k-2}(S_3(n)) + S_{k-3}(S_3(n)) = S_{k-1}(S_3(n))$$

and the theorem is proved.

An immediate consequence of Theorem 6 is

$$\{S_2(n)\} \supseteq \{S_4(n)\} \supseteq \{S_6(n)\} \supseteq \{S_8(n)\} \supseteq \cdots \tag{11}$$

and

$$\{S_3(n)\} \supseteq \{S_5(n)\} \supseteq \{S_7(n)\} \supseteq \{S_9(n)\} \supseteq \cdots . \tag{12}$$

By the theorem of Skolem and Bang, we have

$$\{S_2(n)\}_{n=1}^{\infty} \cap \{S_3(n)\}_{n=1}^{\infty} = \emptyset.$$

Using this result and Theorem 5, it is easy to see that

$$\{S_2(-n)\}_{n=1}^{\infty} \cap \{S_3(-n)\}_{n=1}^{\infty} = \emptyset.$$

Hence $\{S_2(n)\} \cap \{S_3(n)\} = \emptyset$ and $\{S_m(n)\} \cap \{S_{m-1}(n)\} = \emptyset$ for all $m \ge 3$. That is, no row has any elements in common with the row immediately preceding it.

We now turn our attention to an investigation of the columns of Table 1. To do this, we use $C_i$ to represent the $i$th column. You will, after extending the number of columns, see that

$$C_1 \supseteq C_2 \supseteq C_5 \supseteq C_{13} \supseteq C_{34} \cdots$$
$$C_3 \supseteq C_7 \supseteq C_{18} \supseteq C_{47} \supseteq C_{123} \cdots$$

and

$$C_4 \supseteq C_{10} \supseteq C_{26} \supseteq C_{68} \supseteq C_{178} \cdots .$$

Analyzing the subscripts, we are led to conjecture that, for all integers $n \ne 0$,

$$C_{S_1(n)} \supseteq C_{S_3(n)} \supseteq C_{S_5(n)} \supseteq C_{S_7(n)} \supseteq C_{S_9(n)} \supseteq \cdots . \tag{13}$$

In proving this, we arrived at what we believe is an interesting commutative property of this set of sequences.

*Theorem 7*: If $m \ge 1$, then $S_3(S_{2m-1}(n)) = S_{2m-1}(S_3(n))$

*Proof*: The theorem is obviously true for $m = 1$ and $m = 2$. Furthermore, by Theorem 6 and the induction hypothesis, we have

$$S_3(S_{2m+1}(n)) = S_3\big(S_{2m-1}(S_3(n))\big) = S_{2m-1}\big(S_3(S_3(n))\big) = S_{2m+1}(S_3(n)),$$

and the theorem is proved.

Since $S_{2m-1}(S_3(n)) = S_1(S_{2m+1}(n))$, we have

$$S_3(S_{2m-1}(n)) = S_1(S_{2m+1}(n)). \tag{14}$$

Furthermore, by Theorems 6 and 7,

$$S_4(S_{2m-1}(n)) = S_2\big(S_3(S_{2m-1}(n))\big) = S_2(S_{2m+1}(n)). \tag{15}$$

Together, (14) and (15) tell us that

(16) $$C_{S_{2m+1}(n)} \subseteq C_{S_{2m-1}(n)}$$

for all $m \geq 1$. This result proves the validity of (13).

We now turn our attention to the sequences $\{S_m^*(n)\}_{m=1}^{\infty}$. The elements for the first seven sequences are given for $-10 \leq n \leq 10$ in Table 2. An examination of this table leads to results that are very similar to those associated with $\{S_m(n)\}_{m=1}^{\infty}$. A number of these proofs are omitted, since they are similar to the proofs of their counterpart theorems.

*TABLE 2*

| $S_m^*(n)$ \ $n$ | -10 | -9 | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 |
|---|---|---|---|---|---|---|---|---|---|---|
| $S_1^*(n)$ | -10 | -9 | -8 | -7 | -6 | -5 | -4 | -3 | -2 | -1 |
| $S_2^*(n)$ | -17 | -15 | -13 | -12 | -10 | -9 | -7 | -5 | -4 | -2 |
| $S_3^*(n)$ | -28 | -25 | -22 | -20 | -17 | -15 | -12 | -9 | -7 | -4 |
| $S_4^*(n)$ | -46 | -41 | -36 | -33 | -28 | -25 | -20 | -15 | -12 | -7 |
| $S_5^*(n)$ | -75 | -67 | -59 | -54 | -46 | -41 | -33 | -25 | -20 | -12 |
| $S_6^*(n)$ | -122 | -109 | -96 | -88 | -75 | -67 | -54 | -41 | -33 | -20 |
| $S_7^*(n)$ | -198 | -177 | -156 | -143 | -122 | -109 | -88 | -67 | -54 | -33 |

| $S_m^*(n)$ \ $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $S_1^*(n)$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $S_2^*(n)$ | 0 | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 12 | 14 | 16 |
| $S_3^*(n)$ | -1 | 1 | 4 | 6 | 9 | 12 | 14 | 17 | 19 | 22 | 25 |
| $S_4^*(n)$ | -2 | 1 | 6 | 9 | 14 | 19 | 22 | 27 | 30 | 35 | 40 |
| $S_5^*(n)$ | -4 | 1 | 9 | 14 | 22 | 30 | 35 | 43 | 48 | 56 | 64 |
| $S_6^*(n)$ | -7 | 1 | 14 | 22 | 35 | 48 | 56 | 69 | 77 | 90 | 103 |
| $S_7^*(n)$ | -12 | 1 | 22 | 35 | 56 | 77 | 90 | 111 | 124 | 145 | 166 |

*Theorem 8:* If $m$ is an integer and $m \geq 1$, then

$$S_m^*(n) = [n\alpha]F_{m-1} + nF_{m-2} - F_m + 1.$$

*Theorem 9:* If $m$ is an integer and $m \geq 1$, $n \neq 0$, then

$$S_m^*(n) + S_m^*(-n) = -F_{m+2} + 2.$$

*Theorem 10:* If $m \geq 2$ is an integer and $n \neq 0$, then

$$S_m^*(n) = S_{m-1}^*(S_2^*(n)).$$

The proof of Theorem 10 is similar to the proof of Theorem 6, except that one needs Theorem 1 to show that $S_3^*(n) = S_2^*(S_2^*(n))$. The rest of the proof is omitted.

An immediate consequence of Theorem 10 is that if we omit the column when $n = 0$, then every row is a subset of every row preceding it. That is,

$$\{S_1^*(n)\} \supseteq \{S_2^*(n)\} \supseteq \{S_3^*(n)\} \supseteq \{S_4^*(n)\} \supseteq \{S_5^*(n)\} \;\ldots, \tag{17}$$

provided $n \neq 0$.

Using an inductive argument similar to that of Theorem 7, one can show

*Theorem 11:* If $m \geq 1$ is an integer and $n \neq 0$,

$$S_2^*(S_m^*(n)) = S_m^*(S_2^*(n)).$$

Combining Theorems 10 and 11, we have

$$S_2^*(S_m^*(n)) = S_m^*(S_2^*(n)) = S_{m+1}^*(n) = S_1^*(S_{m+1}^*(n)),\; n \neq 0, \tag{18}$$

and

$$S_3^*(S_m^*(n)) = S_2^*\big(S_2^*(S_m^*(n))\big) = S_2^*\big(S_m^*(S_2^*(n))\big) = S_2^*(S_{m+1}^*(n)),\; n \neq 0. \tag{19}$$

Together, (18) and (19) yield

$$C^*_{S^*_{m+1}(n)} \subseteq C^*_{S^*_m(n)} \tag{20}$$

for all integers $m \geq 1$, $n \neq 0$, where $C_i^*$ is the $i$th column of Table 2.

The next result, whose proof we omit, since it is by mathematical induction, establishes a relationship between Table 1 and Table 2.

*Theorem 12:* If $m$ is an integer, $m \geq 1$, $n \neq 0$, then $S_m^*(n) = S_m(n) - F_m + 1$.

Using the fact that $S_2^*(n) = S_2(n)$ in Theorem 10 and applying Theorem 12, we have

$$S_{m+1}(n) + 1 - F_{m+1} = S_{m+1}^*(n) = S_m^*(S_2^*(n)) = S_m(S_2(n)) - F_m + 1$$

or

$$S_{m+1}(n) = S_m(S_2(n)) + F_{m-1},\; n \neq 0. \tag{21}$$

## REFERENCES


1. V. E. Hoggatt, Jr. & A. P. Hillman. "A Property of Wythoff Pairs." *The Fibonacci Quarterly* 16, No. 5 (1978):472.
2. Ivan Niven. *Diophantine Approximations*. Tracts in Mathematics, #14. New York: Interscience Publishers of Wiley & Sons, Inc., 1963, p. 34.
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# THE APOLLONIUS PROBLEM


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On p. 326 of *The Fibonacci Quarterly* 12, No. 4 (1974), Charles W. Trigg gave a formula for the radius of a circle which touches three given circles which, in turn, touch each other externally.

The following is a more general formula: