SOME REMARKS ON THE BELL NUMBERS

LEONARD CARLITZ
Duke University, Durham, NC 27706

1. The Bell numbers A_n can be defined by means of the generating function,

$$e^{e^x - 1} = \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}. \tag{1.1}$$

This is equivalent to

$$A_{n+1} = \sum_{k=0}^{n} \binom{n}{k} A_k. \tag{1.2}$$

Another familiar representation is

$$A_n = \sum_{k=0}^{n} S(n,k), \tag{1.3}$$

where $S(n,k)$ denotes a Stirling number of the second kind [3, Ch. 2].

The definition (1.1) suggests putting

$$e^{a(e^x - 1)} = \sum_{n=0}^{\infty} A_n(a) \frac{x^n}{n!}; \tag{1.4}$$

$A_n(a)$ is called the *single-variable Bell polynomial*. It satisfies the relations

$$A_{n+1}(a) = a \sum_{k=0}^{n} \binom{n}{k} A_k(a) \tag{1.5}$$

and

$$A_n(a) = \sum_{k=0}^{n} a^k S(n,k). \tag{1.6}$$

(We have used A_n and $A_n(a)$ to denote the Bell numbers and polynomials rather than B_n and $B_n(a)$ to avoid possible confusion with Bernoulli numbers and polynomials [2, Ch. 2].)

Cohn, Ever, Menger, and Hooper [1] have introduced a scheme to facilitate the computation of the A_n. See also [5] for a variant of the method. Consider the following array, which is taken from [1].

$A_{n,k}$:

$n \backslash k$	0	1	2	3	4	5	6
0	1	1	2	5	15	52	203
1	2	3	7	20	67	255	1080
2	5	10	27	87	322	1335	
3	15	37	114	409	1657		
4	52	151	523	2066			
5	203	674	2589				
6	877	3263					

The $A_{n,k}$ are defined by means of the recurrence

(1.7) $$A_{n+1,k} = A_{n,k} + A_{n,k+1} \qquad (n \geq 0)$$

together with $A_{00} = 1$, $A_{01} = 1$. It follows that

(1.8) $$A_{0,k} = A_k, \; A_{n,0} = A_{k+1}.$$

The definition of $A_n(a)$ suggests that we define the polynomial $A_{n,k}(a)$ by means of

(1.9) $$A_{n+1,k}(a) = A_{n,k}(a) + A_{n,k+1}(a) \qquad (n \geq 0)$$

together with

$$A_{00}(a) = 1, \; A_{01}(a) = a.$$

We then have

(1.10) $$A_{0,k}(0) = A_k(a), \; aA_{n,0}(a) = A_{n+1}(a).$$

For $a = 1$, (1.10) evidently reduces to (1.8).

2. Put

(2.1) $$F_n(z) = \sum_{k=0}^{\infty} A_{n,k}\frac{z}{k!}$$

and

(2.2) $$F(x,z) = \sum_{n=0}^{\infty} F_n(z)\frac{x^n}{n!} = \sum_{n,k=0}^{\infty} A_{n,k}\frac{x^n z^k}{n!k!}.$$

It follows from (2.1) and the recurrence (1.7) that

(2.3) $$F_{n+1}(z) = F_n(z) + F_n'(z).$$

It is convenient to write (2.3) in the operational form

(2.4) $$F_{n+1}(z) = (1 + D_z)F_n(z) \qquad \left(D_z \equiv \frac{d}{dz}\right).$$

Iteration leads to

(2.5) $$F_n(z) = (1 + D_z)^n F_0(z) \qquad (n \geq 0).$$

Since, by (1.1) and (1.8), $F_0(z) = e^{e^z - 1}$, we get

(2.6) $$F_0(z) = (1 + D_z)^n e^{e^z - 1}.$$

Incidentally, (2.5) is equivalent to

(2.7) $$A_{n,k} = \sum_{j=0}^{n}\binom{n}{j}A_{j+k} = \sum_{j=0}^{n}\binom{n}{j}A_{k+n-j}.$$

The inverse of (2.7) may be noted:

(2.8) $$A_{n+k} = \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}A_{j,k}.$$

Making use of (2.5), we are led to a definition of $A_{n,k}$ for negative n. Replacing n by $-n$, (2.5) becomes

$$(1 + D_z)^n F_{-n}(z) = F_0(z).$$

Thus, if we put

$$F_{-n}(z) = \sum_{k=n}^{\infty} A_{-n,k}\frac{z^k}{k!}, \tag{2.9}$$

we have

$$\sum_{j=0}^{n}\binom{n}{j}A_{-n,j+k} = A_k \qquad (k = 0, 1, 2, \ldots). \tag{2.10}$$

It can be verified that (2.10) is satisfied by

$$A_{-n,k} = \sum_{j=0}^{k-n}(-1)^j\binom{j+n-1}{j}A_{k-n-j} = \sum_{j=0}^{k-n}\binom{-n}{j}A_{k-n-j}. \tag{2.11}$$

Indeed, it is enough to take

$$\begin{aligned} A_{-n,k} + A_{-n,k+1} &= \sum_{j=0}^{k-n}(-1)^j\binom{j+n-1}{j}A_{k-n-j} + \sum_{j=0}^{k-n+1}(-1)^j\binom{j+n-1}{j}A_{k-n-j+1} \\ &= \sum_{j}(-1)^j A_{j-n-j+1}\left\{\binom{j+n-1}{j} - \binom{j+n-2}{j-1}\right\} \\ &= \sum_{j=0}^{k-n+1}(-1)^j\binom{j+n-2}{j}A_{k-n-j+1}, \end{aligned}$$

so that

$$A_{-n,k} + A_{-n,k+1} = A_{-n+1,k} \tag{2.12}$$

and (2.10) follows by induction on n.

Note that by (2.9)

$$A_{-n,k} = 0 \qquad (0 \le k < n). \tag{2.13}$$

The following table of values of $A_{-n,k}$ is computed by means of (2.12) and (2.13).

Put

6	0	0	0	0	0	0	1	-5
5	0	0	0	0	0	1	-4	12
4	0	0	0	0	1	-3	8	-13
3	0	0	0	1	-2	5	-5	54
2	0	0	1	-1	3	0	49	105
1	0	1	0	2	3	49	154	723
0	1	1	2	5	52	203	877	4140
n / k	0	1	2	3	4	5	6	7

Clearly,

$$A_{-n,n} = 1 \qquad (n = 0, 1, 2, \ldots). \tag{2.14}$$

Put

$$G \equiv G(x,z) = \sum_{n=0}^{\infty} F\ (z)x = \sum_{k=0}^{\infty}\frac{z}{n!}\sum_{n=0}^{k}A_{-n,k}x^n.$$

Then, since by (2.12),

$$(1 + D_z)F_{-n}(z) = F_{-n-1}(z) \qquad (n > 0),$$

we have

$$(1 + D_z)G = xG + F_1(z);$$

that is,

$$D\ G + (-x)G = F_1(z) = (1 + e^z)e^{e^z - 1}.$$

This differential equation has the solution

$$e^{(1-x)z}G = \int_0^z e^{(1-x)}\ (1 + e^t)e^{e^t - 1}\,dt + \phi(x), \tag{2.15}$$

where $\phi(x)$ is independent of z.

For $z = 0$, (2.15) reduces to

$$G(x,0) = \phi(x).$$

By (2.15)

$$G(x,0) = A_{0,0} = 1$$

and, therefore

$$G(x,z) = e^{(-1-x)z}\int_0^z e^{(1-x)t}(1 + e^t)e^{e^t - 1}dt + e^{-(1-x)z}. \tag{2.16}$$

In the next place, by (2.2) and (2.5),

$$F(x,z) = \sum_{n=0}^{\infty} \frac{x^n(1 + D_z)^n}{n!}F_0(z) = e^{x(1+D_z)}F_0(z).$$

Since

$$e^{xD_z}F_0(z) = F_0(x + z),$$

we get

$$F(x,z) = e^x F_0(x + z) = e^x e^{e^{x+z} - 1}. \tag{2.17}$$

It follows from (2.5) that

$$e^z F(x,z) = e^x F(z,x), \tag{2.18}$$

which is equivalent to

$$\sum_{j=0}^{k} \binom{k}{j} A_{n,j} = \sum_{j=0}^{n} \binom{n}{j} A_{k,j}. \tag{2.19}$$

Using (2.7), it is easy to give a direct proof of (2.10).

3. The results of §2 are easily carried over to the polynomial $A_n(a)$. Put

$$F_n(z|a) = \sum_{k=0}^{\infty} A_k(a)\frac{z^k}{k!}, \tag{3.1}$$

and

$$F(x,z|a) = \sum_{n=0}^{\infty} F_n(z|a)\frac{x^n}{n!}. \tag{3.2}$$

It follows from (1.9) and (3.1) that

$$F_{n+1}(z|a) = (1 + D_z)F_n(z|a), \tag{3.3}$$

so that

$$F_n(z|a) = (1 + D_z)^n F_0(z|a) = (1 + D_z)^n e^{a(e^z - 1)}. \tag{3.4}$$

Thus,

(3.5) $$A_{n,k}(a) = \sum_{j=0}^{n}\binom{n}{j}A_{j+k}(a).$$

As in §2, we find that

(3.6) $$F(x,z|a) = e^{x}F_0(x+z|a),$$

so that

(3.7) $$e^{z}F(x,z|a) = e^{x}F(z,x|a),$$

which is equivalent to

(3.8) $$\sum_{j=0}^{n}\binom{k}{j}A_{n,j} = \sum_{j=0}^{n}\binom{n}{j}A_{j,k}.$$

By (1.4),

$$\sum_{k=0}^{\infty}A_k(a)\frac{x^k}{k!} = e^{a(e^x-1)}.$$

Thus (3.6) becomes

(3.9) $$F(x,z|a) = e^{x}e^{a(e^{x+z}-1)}.$$

Differentiation with respect to a yields

$$\sum_{n,k=0}^{\infty}A'_{n,k}(a)\frac{x^n z^k}{n!k!} = (e^{x+z}-1)\sum_{n,k=0}^{\infty}A_{n,k}(a)\frac{x^n z^k}{n!k!}$$

and therefore

(3.10) $$A'_{n,k}(a) = \sum_{\substack{i=0\\ i+j<n+k}}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}A_{i,j}(a).$$

Similarly, differentiation with respect to z gives

$$\sum_{n,k=0}^{\infty}A_{n,k+1}(a)\frac{x^n z^k}{n!k!} = ae^{x+y}\sum_{n,k=0}^{\infty}A_{n,k}(a)\frac{x^n z^k}{n!k!},$$

so that

(3.11) $$A_{n,k+1}(a) = a\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}A_{i,j}(a).$$

Comparing (3.11) with (3.10), we get

(3.12) $$A_{n,k+1}(a) = aA_{n,k}(a) + A'_{n,k}(a).$$

Differentiation of (3.9) with respect to x leads again to (1.9).

4. It follows from (1.3) and (2.7) that

(4.1) $$A_{n,k} = \sum_{i=0}^{n}\binom{n}{i}A_{k+i} = \sum_{i=0}^{n}\binom{n}{i}\sum_{j=0}^{k+i}S(k+i,j).$$

Since

$$S(n,j) = \frac{1}{j!}\sum_{t=0}^{j}(-1)^{j-t}\binom{j}{t}t^{k+i},$$

it follows from (4.1) that

$$(4.2) \qquad A_{n,k} = \sum_{j=0}^{k+n} S(n,k,j),$$

where

$$(4.3) \qquad S(n,k,j) = \frac{1}{j!}\sum_{t=0}^{j}(-1)^{j-t}\binom{j}{t} t^k (t+1)^n.$$

Clearly, $S(0,k,j) = S(k,j)$.

In the next place, by (4.1) or (4.3), we have

$$(4.4) \qquad \sum_{k,n=0}^{\infty} S(n,k,j)\frac{x^k y^n}{k!n!} = \frac{e^y}{j!}(e^{x+y} - 1) \ .$$

Differentiation with respect to x gives

$$\sum_{k,n=0}^{\infty} S(n,k+1,j)\frac{x^k y^n}{k!n!} = e^{x+y} \cdot \frac{e^y}{(j-1)!}(e^{x+y} - 1)^{j-1}$$

$$= \frac{e^y}{(j-1)!}(e^{x+y} - 1)^j + \frac{e^y}{(j-1)!}(e^{x+y} - 1)^{j-1},$$

so that

$$(4.5) \qquad S(n,k+1,j) = S(n,k,j-1) + jS(n,k,j),$$

generalizing the familiar formula

$$S(k+1,j) = S(k,j-1) + jS(k,j).$$

Differentiation of (4.4) with respect to x gives

$$\sum_{k,n=0}^{\infty} S(n+1,k,j) = \frac{e^y}{j!}(e^{x+y} - 1)^j + e^{x+y} \cdot \frac{e^y}{(j-1)!}(e^{x+y} - 1)^{j-1}$$

and, therefore

$$(4.6) \qquad S(n+1,k,j) = S(n,k,j) + S(n,k+1,j).$$

This result can be expressed in the form

$$(4.7) \qquad \Delta_n S(n,k,j) = S(n,k+1,j),$$

where Δ_n is the partial difference operator. We can also view (4.6) as the analog of (1.7) for $S(k,n,j)$.

Since $S(0,k,j) = S(k,j)$, iteration of (4.6) yields

$$(4.8) \qquad S(n,k,j) = \sum_{i=0}^{n}\binom{n}{i} S(k+i,j).$$

We recall that

$$x^k = \sum_{j=0}^{k} S(k,j)x(x-1) \ \ldots \ (x-j+1).$$

Hence, it follows from (4.8) that

$$(4.9) \qquad (x+1)^n x^k = \sum_{j=0}^{n+k} S(n,k,j)x(x-1) \ \ldots \ (x-j+1).$$

Replacing x by $-x$, (4.9) becomes

$$(x-1)^n x^k = \sum_{j=0}^{n+k} (-1)^{n+k-j} S(n,k,j) x(x+1) \ldots (x+j-1). \tag{4.10}$$

5. To get a combinatorial interpretation of $A_{n,k}$, we recall [4] that A_k is equal to the number of partitions of a set of cardinality n. It is helpful to sketch the proof of this result.

Let $\overline{A}_k$ denote the number of partitions of the set $S_k = \{1, 2, \ldots, k\}$, $k = 1, 2, 3, \ldots$, and put $\overline{A}_0 = 1$. Then $\overline{A}_{k+1}$ satisfies

$$\overline{A}_{k+1} = \sum_{j=0} \binom{k}{j} \overline{A}_j, \tag{5.1}$$

since the right member enumerates the number of partitions of the set S_{k+1}, as the element $k+1$ is in a block with $0, 1, 2, \ldots, k$ additional elements. Hence, by (1.2),

$$\overline{A}_k = A_k \qquad (k = 0, 1, 2, \ldots).$$

For $A_{n,k}$ we have the following combinatorial interpretation.

Theorem 1: Put $S = \{1, 2, \ldots, n\}$, $T = \{n+1, n+2, \ldots, n+k\}$. Then, $A_{n,k}$ is equal to the number of partitions of all sets $R \cup T$ as R runs through the subsets (the null set included) of S.

The proof is similar to the proof of (5.1), but makes use of (2.7), that is

$$A_{n,k} = \sum_{j=0}^{n} \binom{n}{j} A_{j+k}. \tag{5.2}$$

It suffices to observe that the right-hand side of (5.2) enumerates the partitions of all sets obtained as union of T and the various subsets of S.

For $n = 0$, it is clear that (5.2) gives A_k; for $k = 0$, we get A_{n+1}.

The Stirling number $S(k,j)$ is equal to the number of partitions of the set $1, 2, \ldots, k$ into j nonempty sets. The result for $S(n,k,j)$ that corresponds to Theorem 1 is the following.

Theorem 2: Put $S = \{1, 2, \ldots, n\}$, $T = \{n+1, n+2, \ldots, n+k\}$. Then, $S(n,k,j)$ is equal to the number of partitions into j blocks of all sets $R \cup T$ as R runs through the subsets (the null set included) of S.

The proof is similar to the proof of Theorem 1, but makes use of (4.8), that is,

$$S(n,k,j) = \sum_{i=0}^{n} \binom{n}{i} S(k+i,j). \tag{5.3}$$

REFERENCES

1. M. Cohn, S. Even, K. Menger, & P. K. Hooper. "On the Number of Partitions of a Set of n Distinct Objects." *American Math. Monthly* 69 (1962): 782-785.
2. N.E. Nörlund. *Vorlesungen über Differenzenrechnung*. Berlin: Springer, 1924.
3. J. Riordan. *An Introduction to Combinatorial Analysis*. New York: John Wiley & Sons, Inc., 1958.
4. G.-C. Rota. "The Number of Partitions of a Set." *American Math. Monthly* 71 (1964):498-504.

5. J. Shallit. "A Triangle for the Bell Numbers." *The Fibonacci Quarterly*, to appear.

*****

SOME LACUNARY RECURRENCE RELATIONS

A. G. SHANNON
The New South Wales Institute of Technology, Sydney, Australia
and
Oxford University, Linacre College, England

1. INTRODUCTION

Kirkpatrick [4] has discussed aspects of linear recurrence relations which skip terms in a Fibonacci context. Such recurrence relations are called "lacunary" because there are gaps in them where they skip terms. In the same issue of this journal, Berzsenyi [1] posed a problem, a solution of which is also a lacunary recurrence relation. These are two instances of a not infrequent occurrence.

We consider here some lacunary recurrence relations associated with sequences $\{w_n^{(r)}\}$, the elements of which satisfy the linear homogeneous recurrence relation of order r:

$$w_n^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, w_{n-j}^{(r)},\ n > r,$$

with suitable initial conditions, where the P_{rj} are arbitrary integers. The sequence, $\{v_n^{(r)}\}$, with initial conditions given by

$$v_n^{(r)} = \begin{cases} 0 & n < 0, \\ \sum_{j=1}^{r} \alpha_{rj}^n & 0 \le n < r \end{cases}$$

is called the "primordial" sequence, because when $r = 2$, it becomes the primordial sequence of Lucas [6]. The α_{rj} are the roots, assumed distinct, of the auxiliary equation

$$x^r = \sum_{j=1}^{r} (-1)^{j+1} P_{rj}\, x^{r-j}.$$

We need an arithmetical function $\delta(m,s)$ defined by

$$\delta(m,s) = \begin{cases} 1 & \text{if } m \mid s, \\ 0 & \text{if } m \nmid s. \end{cases}$$

We also need $s(r,m,j)$, the symmetric functions of the α_{ri}^m, $i = 1, 2, \ldots, r$, taken j at a time, as in Macmahon [5]:

$$s(r,m,j) = \Sigma \alpha_{ri_1}^m \alpha_{ri_2}^m \ldots \alpha_{ri_j}^m,$$

in which the sum is over a distinct cycle of α_{ri}^m taken j at a time and where we set $s(r,m,0) = 1$.