FIBONACCI NOTES

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6. ZERO-ONE SEQUENCE ONCE MORE

1. Let $f(m, n, r, s)$ denote the number of zero-one sequences of length $m+n$:

$$\sigma = (a_1, a_2, \ldots, a_{m+n}) \qquad (a \;= 0 \text{ or } 1) \tag{1.1}$$

with m zeros, n ones, r occurrences of (00), and s occurrences of (11). It is proved in [1] that

$$f(m, n, r, s) = \begin{cases} 2\binom{m-1}{r}\binom{n-1}{s} & (m - r = n - s) \\ \binom{m-1}{r}\binom{n-1}{s} & (m - r = n - s \pm 1) \\ 0 & (\text{otherwise}). \end{cases} \tag{1.2}$$

The proof in [1] makes use of generating functions; we shall now give a combinatorial proof of (1.2).

Arrange the m zeros and n ones in the following way. We first place m_0 zeros on the extreme left, then n_1 ones, m_1 zeros, n_2 ones, n_2 zeros, ..., n_k ones, m_k zeros, where k is some nonnegative integer,

$$\begin{aligned} &m = m_0 + m_1 + \cdots + m_k,\; n = n_1 + \cdots + n_k, \\ &m_0 \geq 0,\; m \;\geq 0,\; m \;\geq 1 \quad (1 \leq i < k) \\ &n_1 \geq 0 \quad (1 \leq i \leq k) \end{aligned} \tag{1.3}$$

and

$$\begin{cases} r = \displaystyle\sum_{i=0}^{k} (m_i - 1) + \delta + \delta' = m - k - 1 + \delta + \delta' \\ s = \displaystyle\sum_{i=1}^{k} (n_i - 1) = n - k, \end{cases} \tag{1.4}$$

where

$$\begin{aligned} \delta &= \begin{cases} 1 & (m_0 = 0) \\ 0 & (m_0 > 0), \end{cases} \\ \delta' &= \begin{cases} 1 & (m_k = 0) \\ 0 & (m_k > 0). \end{cases} \end{aligned} \tag{1.5}$$

It follows from (1.3) and (1.4) that

$$r - s = m - n + \delta + \delta' - 1. \tag{1.6}$$

It is now convenient to consider four cases:

$$\text{(i)}\;\; m_0 = m_k = 0; \qquad \text{(ii)}\;\; m_0 = 0,\; m_k > 0;$$

$$\text{(iii)}\;\; m_0 > 0,\; m_k = 0; \;\; \text{(iv)}\;\; m_0 > 0,\; m_k > 0.$$

The number of solutions of

$$a = x_1 + \cdots + x_k, \; x_i > 0 \quad (i = 1, \ldots, k)$$

is equal to $\binom{a-1}{k-1}$.

Thus, the number of solutions

$$(m_0, m_1, \ldots, m_k; n_1, \ldots, n_k)$$

of (1.3) is equal to:

$$\text{(i)} \qquad \binom{m-1}{k-2}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s - 1)$$

$$\text{(ii)} \qquad \binom{m-1}{k-1}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s)$$

$$\text{(iii)} \qquad \binom{m-1}{k-1}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s)$$

$$\text{(iv)} \qquad \binom{m-1}{k}\binom{n-1}{k-1} = \binom{m-1}{r}\binom{n-1}{s} \quad (m - r = n - s + 1).$$

The first part of (1.2) is implied by (ii) together with (iii), the second part by (i) and (iv). The last part of (1.2) is equivalent to the statement that k cannot exist satisfying both parts of (1.4).

This evidently completes the proof of (1.2).

2. The above proof is applicable to a much more general problem. Let

$$\text{(2.1)} \qquad \boldsymbol{r} = (r_1, r_2, r_3, \ldots), \quad \boldsymbol{s} = (s_1, s_2, s_3, \ldots)$$

be two sequences of nonnegative integers. We again consider zero-one sequences of length $m+n$ with m zeros and n ones. Let $f(\boldsymbol{r}, \boldsymbol{s})$ denote the number of such sequences, where $r_1 = m$, $s_1 = n$, with r_i blocks of zeros of length i and s_i blocks of ones of length i for $i = 2, 3, 4, \ldots$. Thus, r_1 can be thought of as the number of blocks of zeros of length one and s_1 the number of blocks of length one.

As in §1, we envisage an arbitrary sequence σ as broken into a block of zeros (possibly vacuous), a block of ones, a block of zeros, and so on. However, we shall now enumerate the blocks by their cardinality. If k denotes the number of blocks of ones, then the number of blocks of zeros is either $k - 1$, k, or $k + 1$. Hence, we have the following relations,

$$\text{(2.2)} \qquad \begin{cases} r_1 = k_1' + 2k_2' + 3k_3' + \cdots \\ r_2 = k_2' + 2k_3' + 3k_4' + \cdots \\ r_3 = k_3' + 2k_4' + 3k_5' + \cdots \\ \cdots\cdots\cdots\cdots \end{cases}$$

and

$$(2.3)\qquad \begin{cases} s_1 = k_1 + 2k_2 + 3k_3 + \cdots \\ s_2 = k_2 + 2k_3 + 3k_4 + \cdots \\ s_3 = k_3 + 2k_4 + 3k_5 + \cdots \\ \cdots\cdots\cdots\cdots\cdots \end{cases}$$

together with

$$(2.4)\qquad \begin{cases} k' = k_1' + k_2' + k_3' + \cdots \\ k = k_1 + k_2 + k_3 + \cdots, \end{cases}$$

where $k' = k - 1$, k, or $k + 1$.

The k_i' denote the multiplicity of blocks of zeros of length i, and the k_i denote the multiplicity of blocks of ones of length i. Thus, the first of (2.2) enumerates the number of blocks of zeros of length one, that is, the total number of zeros. The second of (2.2) enumerates the number of blocks of zeros of length two, and so on. Similar remarks apply to (2.3) for the blocks of ones.

It is easily verified that (2.2) is equivalent to the system of equations

$$(2.5)\qquad \begin{cases} k_1' = r_1 - 2r_2 + r_3 \\ k_2' = r_2 - 2r_3 + r_4 \\ k_3' = r_3 - 2r_4 + r_5 \\ \cdots\cdots\cdots\cdots \end{cases}$$

while (2.3) is equivalent to

$$(2.6)\qquad \begin{cases} k_1 = s_1 - 2s_2 + s_3 \\ k_2 = s_2 - 2s_3 + s_4 \\ k_3 = s_3 - 2s_4 + s_5 \\ \cdots\cdots\cdots\cdots \end{cases}$$

Thus, the r_i and s_i must satisfy the following conditions, but are otherwise unrestricted.

$$(2.7)\qquad \begin{cases} r_i - 2r_{i+1} + r_{i+2} \geq 0 \\ s_i - 2s_{i+1} + s_{i+2} \geq 0 \end{cases} \qquad (i = 1, 2, 3, \ldots).$$

It follows from (2.5), (2.6), and (2.4) that

$$(2.8)\qquad \begin{cases} k' = r_1 - r_2 \\ k = s_1 - s_2. \end{cases}$$

Clearly,

$$f(\mathbf{r},\ \mathbf{s}) = \frac{k'!}{k_1'!k_2'!k_3!' \ldots} \cdot \frac{k!}{k_1!k_2!k_3! \ldots} \tag{2.9}$$

In terms of r_i and s_i, this becomes

$$f(\mathbf{r},\ \mathbf{s}) = \frac{(r_1 - r_2)!}{(r_1 - 2r_2 + r_3)!(r_2 - 2r_3 + r_4)! \ldots} \cdot \frac{(s_1 - s_2)!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3 + s_4)! \ldots} \tag{2.10}$$

3. For applications, it is convenient to use generating functions. By the multinomial theorem, we have

$$\sum_{k_1+k_2+k_3+\cdots=k} \frac{k!}{k_1!k_2!k_3! \ldots} x_1^{k_1}x_2^{k_2}x_3^{k_3} \ldots = (x_1 + x_2 + x_3 + \cdots)^k, \tag{3.1}$$

where it is assumed that the series $x_1 + x_2 + x_3 + \cdots$ is absolutely convergent. By (2.6), the left-hand side of (3.1) is equal to

$$\sum_{\substack{\mathbf{s} \\ s_1 - s_2 = k}} \frac{k!}{(s_1 - 2s_2 + s_3)!(s_1 - 2s_2 + s_3)! \ldots} x_1^{s_1 - 2s_2 + s_3} x_2^{s_2 - 2s_3 + s_4} \ldots$$

$$= \sum_{\substack{\mathbf{s} \\ s_1 - s_2 = k}} \frac{k!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3 + s_4)! \ldots} x_1^{s_1}(x_1^{-2}x_2)^{s_2} (x_1x_2^{-2}x_3)^{s_3} \ldots .$$

Hence, if we take

$$\begin{cases} x_1 = y_1 \\ x_2 = y_1^2y_2 \\ x_3 = y_1^3y_2^2y_3 \\ x_4 = y_1^4y_2^3y_3^2y_4 \\ \ldots\ldots\ldots \end{cases}$$

(3.1) becomes

$$(y_1 + y_1^2y_2 + y_1^3y_2^2y_3 + \cdots)^k \tag{3.2}$$

$$= \sum_{\substack{\mathbf{s} \\ s_1 - s_2 = k}} \frac{k!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3 + s_4)! \ldots} y_1^{s_1} y_2^{s_2} y_3^{s_3} \ldots .$$

As a first application of (3.2), we take $y_3 = y_4 = y_5 = \cdots = 1$. Then, the left-hand side of (4.2) reduces to

$$\begin{aligned}(y_1 + y_1^2 y_2 + y_1^3 y_2^2 + \cdots)^k &= y_1^k (1 - y_1 y_2)^{-k} \\ &= \sum_{s=0}^{\infty} \binom{k + s - 1}{s} y_1^{s+k} y_2^s \\ &= \sum_{s_1 - s_2 = k} \binom{s_1 - 1}{s_2} y_1^{s_1} y_2^{s_2},\end{aligned}$$

in agreement with (1.2).

If we take $y_3 = y_4 = \cdots = 0$, we get

$$\begin{aligned}(y_1 + y_1^2 y_2)^k &= y_1^k \sum_{s=0}^{\infty} \binom{k}{s} y_2^s \\ &= \sum_{s_1 - s_2 = k} \binom{s_1 - s_2}{s_2} y_1^{s_1} y_2^{s_2}.\end{aligned}$$

Thus, in this case, we have

$$f(\boldsymbol{r}, \boldsymbol{s}) = \binom{r_1 - r_2}{r_2}\binom{s_1 - s_2}{s_2}, \tag{3.3}$$

where $r_1 - r_2 = k'$, $s_1 - s_2 = k$, while

$$r_3 = r_4 = \cdots = 0, \quad s_3 = s_4 = \cdots = 0.$$

That is, (3.3) furnishes the enumerant when all blocks are of length one or two.

4. In (3.2), we now take

$$y_4 = y_5 = y_6 = \cdots = 1. \tag{4.1}$$

Then, the left-hand side of (3.2) becomes

$$\begin{aligned}&(y_1 + y_1^2 y_2 + y_1^3 y_2^2 y_3 + y_1^4 y_2^3 y_3^2 + \cdots)^k \\ &= y_1^k \left\{1 + y_1 y_2 (1 + y_1 y_2 y_3 + y_1^2 y_2^2 y_3^2 + \cdots)\right\}^k \\ &= y_1^k \left\{1 + \frac{y_1 y_2}{1 - y_1 y_2 y_3}\right\}^k \\ &= y_1^k \sum_{t=0}^{k} {k \atop t} \; (y_1 y_2) \; \sum_{s=0}^{\infty} \binom{t + s - 1}{s} (y_1 y_2 y_3) \\ &= \sum_{\substack{s_1, s_2, s_3 \\ s_1 - s_2 = k}} \binom{s_1 - s_2}{s_2 - s_3} \binom{s_2 - 1}{s_3} y_1^{s_1} y_2^{s_2} y_3^{s_3}.\end{aligned}$$

Hence, we have

$$(4.2)\qquad f(\boldsymbol{r}, \boldsymbol{s}) = \binom{r_1 - r_2}{r_2 - r_3}\binom{r_2 - 1}{r_3}\binom{s_1 - s_2}{s_2 - s_3}\binom{s_2 - 1}{s_3},$$

where $r_1 - r_2 = k'$, $s_1 - s_2 = k$.

Thus (4.2) furnishes the enumerant by blocks of length 1, 2, and 3.

If, instead of (4.1), we take

$$(4.3)\qquad y_4 = y_5 = y_6 = \cdots = 0,$$

we have

$$(y_1 + y_1^2y_2 + y_1^3y_2^2y_3)^k = \sum_{t_1+t_2+t_3=k} \frac{k!}{t_1!t_2!t_3!} y_1^{t_1+2t_2+t_3} y_2^{t_2+2t_3} y_3^{t_3}$$

$$= \sum_{\substack{s_1, s_2, s_3 \\ s_1 - s_2 = k}} \frac{(s_1 - s_2)!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3)!s_3!} y_1^{s_1} y_2^{s_2} y_3^{s_3}$$

so that

$$(4.4)\qquad f(\boldsymbol{r}, \boldsymbol{s}) = \frac{(r_1 - r_2)!}{(r_1 - 2r_2 + r_3)!(r_2 - 2r_3)!r_3!} \cdot \frac{(s_1 - s_2)!}{(s_1 - 2s_2 + s_3)!(s_2 - 2s_3)!s_3!},$$

the enumerant when *all* blocks are of length 1, 2, or 3.

5. The general cases corresponding to (4.2) and (4.4) are now readily obtained. Let p be a fixed positive integer, and take

$$(5.1)\qquad y_{p+1} = y_{p+2} = \cdots = 1.$$

Then we have

$$(5.2)\qquad \left\{ y_1 + y_1^2y_2 + \cdots + y_1^{p-2}y_2^{p-3} \cdots y_{p-2} + \frac{y_1^{p-1}y_2^{p-2} \cdots y_{p-1}}{1 - y_1y_2 \cdots y_p} \right\}^k$$

$$= \sum_{t_1+\cdots+t_{p-1}=k} (t_1, t_2, \ldots, t_{p-1}) y_1^{t_1'} y_2^{t_2'} \cdots y_{p-1}^{t_{p-1}'} \sum_{s=0}^{\infty} \binom{t_{p-1} + s - 1}{s} (y_1y_2 \cdots y_p)^s,$$

where

$$(t_1, t_2, \ldots, t_{p-1}) = \frac{(t_1 + t_2 + \cdots + t_{p-1})!}{t_1!t_2! \ldots t_{p-1}!}$$

and

$$\begin{cases} t_1' = t_1 + 2t_2 + \cdots + (p-1)t_{p-1} \\ t_2' = t_2 + 2t_3 + \cdots + (p-2)t_{p-2} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ t_{p-2}' = t_{p-2} + 2t_{p-1} \\ t_{p-1}' = t_{p-1}. \end{cases}$$

Put

$$t_i' + s = s_i \quad (1 \le i < p), \quad s = s_p.$$

It follows that

$$\begin{cases} t_{p-1} = s_{p-1} - s_p \\ t_{p-2} = s_{p-2} - 2s_{p-1} + s_p \\ t_{p-3} = s_{p-3} - 2s_{p-2} + s_{p-1} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ t_1 = s_1 - 2s_2 + s_3. \end{cases} \tag{5.3}$$

Hence, the coefficient of $y_1^{s_1}y_2^{s_2} \ldots y_p^{s_p}$ in (5.2) is equal to

$$(t_1, t_2, \ldots, t_{p-1})\binom{s_{p-1} - 1}{s}, \tag{5.4}$$

where $t_1, t_2, \ldots, t_{p-1}$ are given by (5.3).

The enumerant $f(\mathbf{r}, \mathbf{s})$ is therefore equal to (5.4) times the corresponding factor containing the r_i.

Corresponding to

$$y_{p+1} = y_{p-2} = \cdots = 0, \tag{5.5}$$

we have

$$(y_1 + y_1^2y_2 + \cdots + y_1^py_2^{p-1} \cdots y_p)^k$$
$$= \sum_{t_1 + \cdots + t_p = k} (t_1, t_2, \ldots, t_p)y_1^{s_1}y_2^{s_2} \ldots y_p^{s_p}, \tag{5.6}$$

where now

$$\begin{cases} t_1 + 2t_2 + 3t_3 + \cdots + pt_p = s_1 \\ t_2 + 2t_3 + 3t_4 + \cdots + (p-1)t_p = s_2 \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ t_{p-1} + 2t_p = s_{p-1} \\ t_p = s_p. \end{cases}$$

This gives

$$(5.7)\qquad \begin{cases} t_p = s_p \\ t_{p-1} = s_{p-1} - 2s_p \\ t_{p-2} = s_{p-2} - 2s_{p-1} + s_p \\ t_{p-3} = s_{p-3} - 2s_{p-2} + s_{p-1} \\ \cdots\cdots\cdots\cdots\cdots \\ t_1 = s_1 - 2s_2 + s_3. \end{cases}$$

Hence, the coefficient of $y_1^{s_1}y_2^{s_2} \ldots y_p^{s_p}$ is the multinomial coefficient $(t_1, t_2, \ldots, t_p)$, with the t_i determined by (5.7). The enumerant $f(\boldsymbol{r}, \boldsymbol{s})$ is the product of this coefficient times the corresponding factor containing the r_i.

6. Some curious combinatorial identities are implied by the above results. To illustrate with a simple case, we return to §3. It follows from (3.1) that, for $s_1 > s_2$, we have

$$(6.1)\qquad \sum(t_1, t_2, t_3, \ldots) = \binom{s_1 - 1}{s_2},$$

where

$$t_i = s_i - 2s_{i+1} + s_{i+2} \quad (i = 1, 2, 3, \ldots),$$

and the summation is over all $s_3, s_4, s_5, \ldots$.

Similarly, from the proof of (4.2), we have, for

$$s_1 - 2s_2 + s_3 \geq 0, \quad s_2 > s_3,$$

$$(6.2)\qquad \sum(t_1, t_2, t_3, \ldots) = \binom{s_1 - s_2}{s_2 - s_3}\binom{s_2 - 1}{s_3},$$

where

$$t_i = s_i - 2s_{i+1} + s_{i+2} \quad (i = 1, 2, 3, \ldots),$$

and the summation is over all $s_4, s_5, s_6, \ldots$.

The general case implied by (5.2) and (5.4) is readily stated. We have

$$(6.3)\qquad \sum(t_1, t_2, t_3, \ldots) = (\overline{t}_1, \overline{t}_2, \ldots, \overline{t}_{p-1})\binom{s_{p-1} - 1}{s_p},$$

where

$$t_i = s_i - 2s_{i+1} + s_{i+2} \quad (i = 1, 2, 3, \ldots)$$

$$\overline{t}_i = t_i \quad (i = 1, \ldots, p - 2), \quad \overline{t}_{p-1} = s_{p-1} - s_p,$$

and the summation on the left of (6.3) is over all $s_{p+1}, s_{p+2}, s_{p+3}, \ldots$.

There are various other possibilities; for example, taking $y = 1$ in (3.2). However, we leave this for another occasion.

REFERENCE

1. L. Carlitz. "Fibonacci Notes: 5. Zero-One Sequences Again." *The Fibonacci Quarterly* 15 (1977):49-56.

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