ELEMENTARY PROBLEMS AND SOLUTIONS

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Send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, New Mexico 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within 4 months of the publication date.

DEFINITIONS

The Fibonacci numbers F_n and Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$. Also a and b designate the roots $(1 + \sqrt{5})/2$ and $(1 - \sqrt{5})/2$, respectively, of $x^2 - x - 1 = 0$.

PROBLEMS PROPOSED IN THIS ISSUE

B-424 *Proposed by Richard M. Grassl, University of New Mexico*

Of the $\binom{52}{5}$ possible 5-card poker hands, how many form a:

(i) full house?
(ii) flush?
(iii) straight?

B-425 *Proposed by Richard M. Grassl, University of New Mexico*

Let k and n be positive integers with $k < n$ and let S consist of all k-tuples $X = (x_1, x_2, \ldots, x_k)$ with each x_j an integer and

$$1 \le x_1 < x_2 < \cdots < x_k \le n.$$

For $j = 1, 2, \ldots, k$, find the average value $\overline{x}_j$ of x_j over all X in S.

B-426 *Proposed by Herta T. Freitag, Roanoke, VA*

Is $(F_nF_{n+3})^2 + (2F_{n+1}F_{n+2})^2$ a perfect square for all positive integers n, i.e., are there integers c_n such that $(F_nF_{n+3}, 2F_{n+1}F_{n+2}, c_n)$ is always a Pythagorean triple?

B-427 *Proposed by Phil Mana, Albuquerque, NM*

Establish a closed form for $\sum_{k=1}^{n} k\binom{k}{2}\binom{n-k}{3}$.

B-428 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

For odd positive integers w, establish a closed form for

$$\sum_{k=0}^{2s+1} \binom{2s+1}{k} F^2_{n+kw}.$$

B-429 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Is the function

$$F^4_{n+10r} + F^4_n - (L_{8r} + L_{4r} - 1)(F^4_{n+8r} + F^4_{n+2r}) + (L_{12r} - L_{8r} + 2)(F^4_{n+6r} + F^4_{n+4r})$$

independent of n? Here n and r are integers.

SOLUTIONS

Multiples of Some Triangular Numbers

B-400 *Proposed by Herta T. Freitag, Roanoke, VA*

Let T_n be the nth triangular number $n(n+1)/2$. For which positive integers n is $T_1^2 + T_2^2 + T_3^2 + \cdots + T_n^2$ an integral multiple of T_n?

Solution by C. C. Thompson, Roanoke, VA

Let $S = \sum_{k=1}^{n} T_n^2$, where n is a positive integer; then S is an integral multiple of T_n iff $n \equiv 1, 7, 13 \pmod{15}$. To see this, use the formulas for sums of powers of the first n positive integers (or the method of differences) and a bit of manipulative algebra to get

$$S = T_n \cdot (3n^3 + 12n^2 + 13n + 2)/30.$$

From this, the sum S is an integral multiple of T_n iff

$$f(n) = 3n^3 + 12n^2 + 13n + 2 \equiv 0 \pmod{2 \cdot 3 \cdot 5}.$$

Now $f(n) \equiv n^3 + n \equiv n(n+1)^2 \equiv 0 \pmod 2$ is satisfied by any positive integer; $f(n) \equiv n + 2 \equiv 0 \pmod 3$ has $n \equiv 1 \pmod 3$ as its only solution; $f(n) \equiv (3n+2)(n^2+1) \equiv 0 \pmod 5$ has $n \equiv 1, 2, 3 \pmod 5$ as solutions. From this, $f(n) \equiv 0 \pmod{30}$ has the solutions $n \equiv 1, 7, 13 \pmod{15}$.

Also solved by Paul S. Bruckman, Edilio A. Escalona Fernández, Bob Prielipp, Sahib Singh, M. Wachtel (Switzerland), Jonathan Weitsman, Gregory Wulczyn, and the proposer.

Change of Pace for F.Q.

B-401 *Proposed by Gary L. Mullen, Pennsylvania State University*

Show that $\lim_{n\to\infty}[(n!)^{2n}/(n^2)!] = 0$.

Solution by Edilio A. Escalona Fernández, Caracas, Venezuela

Let's call $R_n = (n!)^{2n}/(n^2)!$, and $T_n = Ln(R_n)$. Then,

$$T_n = 2nLn(n!) - Ln((n^2)!),$$

so that by applying the formula $Ln(n!) = nLn(n) - n + 0(Ln(n))$, we have

$$T_n = -n^2 + 2n0(Ln(n)) + 0(Ln(n)) = -n^2 + 0(nLn(n)),$$

and this means that $T_n \to -\infty$ as $n \to \infty$; hence, by continuity of $\exp(x)$:

$$\exp(T_n) = R_n \to 0 \text{ as } n \to \infty.$$

Also solved by Paul S. Bruckman, M. Wachtel (Switzerland), Jonathan Weitsman, Gregory Wulczyn, and the proposer.

Pythagorean Triple

B-402 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Show that $(L_nL_{n+3},\ 2L_{n+1}L_{n+2},\ 5F_{2n+3})$ is a Pythagorean triple.

Solution by Sahib Singh, Clarion College, Clarion, PA

Let $A = L_{n+2}$, $B = L_{n+1}$, then

$$A^2 - B^2 = (L_{n+2} - L_{n+1})(L_{n+2} + L_{n+1}) = L_nL_{n+3}.$$
$$A^2 + B^2 = L^2_{n+2} + L^2_{n+1} = 5(F^2_{n+2} + F^2_{n+1}) = 5F_{2n+3}.$$

Thus, the given triple is $A^2 - B^2$, $2AB$, $A^2 + B^2$, which is Pythagorean.

Also solved by Paul S. Bruckman, Herta T. Freitag, Graham Lord, John W. Milsom, Bob Prielipp, and the proposer.

Lucas Congruence

B-403 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Let $m = 5^n$. Show that $L_{2m} \equiv -2 \pmod{5m^2}$.

Solution by Graham Lord, Université Laval, Québec;
Bob Prielipp, University of Wisconsin-Oshkosh; and
Sahib Singh, Clarion College, Clarion, Pa (independently)

It is known that $m|F_m$. [See B-248, vol. 11 (1973):553.] Hence,

$$(5m^2)\,|\,(5F^2_m).$$

Since m is odd, we also have $L_{2m} = 5F^2_m - 2$, and it follows that

$$L_{2m} \equiv -2 \pmod{5m^2}.$$

Also solved by Paul S. Bruckman, Lawrence Somer, and the proposer.

Golden Approximations

B-404 *Proposed by Phil Mana, Albuquerque, NM*

Let x be a positive irrational number. Let a, b, c, and d be positive integers with $a/b < x < c/d$. If $a/b < r < x$, with r rational, implies that the denominator of r exceeds b, we call a/b a good lower approximation (GLA) for x. If $x < r < c/d$, with r rational, implies that the denominator of r exceeds d, c/d is a good upper approximation (GUA) for x. Find all the GLAs and all the GUAs for $(1 + \sqrt{5})/2$.

Solution by Paul S. Bruckman, Concord, CA

Let

(1) $$x_n = F_{2n}/F_{2n-1},\ y_n = F_{2n+1}/F_{2n},\ n = 1, 2, 3, \ldots;$$

let

(2) $$X = (x_n)^\infty_{n=1},\ Y = (y_n)^\infty_{n=1}.$$

It is well known that X and Y provide the convergents for the continued fraction of α, and moreover:

(3) $1 = x_1 < x_2 < \cdots < x_n < \cdots \alpha \cdots < y_n < \cdots < y_2 < y_1 = 2.$

Let L and U denote the set of GLAs and GUAs, respectively, for α. We will prove that

(4) $$L = X, \; U = Y.$$

We will use the following result, readily proved by applying the Binét definitions:

(5) $$F_{2n+2}F_{2n-1} - F_{2n}F_{2n+1} = 1.$$

Proof of (4): Given any positive integer n, and any rational $r = u/v$, such that $x_n < r \leq x_{n+1}$, then, $x_{n+1} - x_n \geq r - x_n > 0$, i.e.,

$$\frac{F_{2n+2}}{F_{2n+1}} - \frac{F_{2n}}{F_{2n-1}} \geq \frac{u}{v} - \frac{F_{2n}}{F_{2n-1}} > 0$$

$$\Rightarrow v(F_{2n+2}F_{2n-1} - F_{2n}F_{2n+1}) \geq F_{2n+1}(uF_{2n-1} - vF_{2n}) > 0.$$

But, since $u/v > F_{2n}/F_{2n-1}$, thus $uF_{2n-1} - vF_{2n} \geq 1$; using (5), this implies

(6) $$v \geq F_{2n+1}.$$

Since $F_{2n-1} < F_{2n+1}$, thus $v > F_{2n-1}$, which implies that $x_n \in L$. Hence,

(7) $$X \subseteq L.$$

Conversely, suppose $r = u/v \in L$. Then, for some n, $x_n < r \leq x_{n+1}$, which again implies (6), as above. Assume that $r < x_{n+1}$. Then, by definition of L, $v < F_{2n+1}$, which contradicts (6). It follows that $r = x_{n+1} \Rightarrow r \in X$. Hence,

(8) $$L \subseteq X.$$

Combining (7) and (8) implies $L = X$. Proceeding in a totally analogous manner, we may likewise prove that $U = Y$.

Also solved by Sahib Singh, Gregory Wulczyn, and the proposer.

Good Rational Approximations

B-405 *Proposed by Phil Mana, Albuquerque, NM*

Prove that for every positive irrational x, the GLAs and GUAs for x (as defined in B-404) can be put together to form one sequence $\{p_n/q_n\}$ with

$$p_{n+1}q_n - p_nq_{n+1} = \pm 1 \text{ for all } n.$$

Solution by the proposer.

Let $p = [x]$, the greatest integer in x. Clearly p is a GLA and $p + 1$ is a GUA. So we let $p_1 = p$, $q_1 = 1 = q_2$, and $p_2 = p + 1$. Then we assume inductively that p_n and q_n have been defined for $n = 1, 2, \ldots, k$. Let s be the largest such n for which p_n/q_n is a GLA and t be the largest such n for which p_n/q_n is a GUA; then define $p_{n+1} = p_s + p_t$ and $q_{n+1} = q_s + q_t$. This defines p_n and q_n for all positive integers n and we let $r = p_n/q_n$. It follows from the theory of Farey sequences [see Ivan Niven & Herbert S. Zuckerman, *An Introduction to the Theory of Numbers* (New York: Wiley, 1960), pp. 128-133) that the r_n give us all the GLAs and GUAs and that $p_{n+1}q_n - p_nq_{n+1} = \pm 1$.

Also solved by Paul S. Bruckman and Sahib Singh.

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