Using (2.4) and (2.5), we see that $a_{01} = 1$ or 2 and $a_{10} = 1$ or 2. From (2.6) and (2.7), we have $a_{11} = 0$. Since a_{00} is arbitrary, we see that there are a total of twelve local permutation polynomials over Z_3, given by

$$f(x_1, x_2) = a_{10}x_1 + a_{01}x_2 + a_{00},$$

where $a_{10} = 1$ or 2, $a_{01} = 1$ or 2, and $a_{00} = 0, 1,$ or 2.

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GENERALIZED CYCLOTOMIC POLYNOMIALS, FIBONACCI CYCLOTOMIC POLYNOMIALS, AND LUCAS CYCLOTOMIC POLYNOMIALS*

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1. INTRODUCTION AND MAIN THEOREM

In [6], Hoggatt and Long ask what polynomials in $I[x]$ are divisors of the Fibonacci polynomials, which are defined by the recursion

$$F_0(x) = 0,\ F_1(x) = 1,\ F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \text{ for } n \geq 2.$$

In this paper, we answer this question in terms of cyclotomic polynomials. We prove that each Fibonacci polynomial $F_n(x)$, for $n \geq 2$, has one and only one irreducible factor which is not a factor of any $F_k(x)$ for any positive k less than n. We call this irreducible factor the nth *Fibonacci cyclotomic polynomial* and denote it $\mathcal{F}_n(x)$.

The method applied to F_n's to produce $\mathcal{F}_n$'s applies naturally to the more general polynomials $\ell_n(x, y, z)$ which were introduced in [7] and are defined just below. Accordingly, in Section 2, we shall apply the method at this more general level rather than directly to the F_n's. The polynomials $C_n(x, y, z)$ so obtained from the $\ell_n(x, y, z)$'s we call *generalized cyclotomic polynomials*. Special cases of the C_n's are the ordinary cyclotomic polynomials $C_n(x, 1, 0)$, the Fibonacci cyclotomic polynomials $\mathcal{F}_n$ already mentioned, and a sequence

$$\mathcal{L}_n(x) = C_n(x, 0, 1)$$

which we call the *Lucas cyclotomic polynomials*. Section 3 is devoted to the $\mathcal{F}_n$'s and Section 4 to the $\mathcal{L}_n$'s. In Sections 3, 4, and 5, we determine all the irreducible factors of the Fibonacci polynomials, the modified Lucas polynomials defined in [7] as $\ell_n(x, 0, 1)$, and the Lucas polynomials.

In Section 6, we transform the generalized Fibonacci and Lucas polynomials into sequences $U_n(x, z)$ and $V_n(x, z)$ having the same divisibility properties as the F_n's and L_n's, respectively. The coefficients of these polynomials are all binomial coefficients, in accord with the identity

$$zU_n(x, z) + V_n(x, z) = (x + z)^n.$$

The polynomials $\ell_n(x, y, z)$ may be defined as follows:

$$\ell_n(x, y, z) = \frac{L_n(x, z) - L_n(y, z)}{x - y} \quad \text{for } n \geq 0,$$

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where $L_n(x, z)$ is the nth generalized Lucas polynomial, defined by the recursion

$$L_0(x, z) = 2,\ L_1(x, z) = x,\ L_n(x, z) = xL_{n-1}(x, z) + zL_{n-2}(x, z) \text{ for } n \geq 2.$$

The two special cases of particular interest are the generalized Fibonacci polynomials, namely

$$\ell_n\left(\frac{x + \sqrt{x^2 + 4z}}{2}, \frac{x - \sqrt{x^2 + 4z}}{2}, 0\right), \tag{1}$$

and the generalized modified Lucas polynomials, namely $\ell_n(x, 0, z)$. Other special cases, to be treated briefly in Section 5, are the Chebyshev polynomials of the first and second kinds.

Following the method of Hoggatt and Bicknell in [5], we now determine the roots of the polynomials $\ell_n(x, y, z)$. The first theorem is basic to all subsequent developments in this paper.

Theorem 1: For $n \geq 2$, the roots of $\ell_n(x, y, z)$ are

$$2\sqrt{z} \sinh (\sinh^{-1} y/2\sqrt{z} + 2k\pi i/n), \text{ where } k = 1, 2, \ldots, n - 1. \tag{2}$$

Proof: We have $(x - y)\ell_n(x, y, z) = t_1^n + t_2^n - (t_3^n + t_4^n)$, where

$$t_1 = \frac{x + \sqrt{x^2 + 4z}}{2},\ t_2 = \frac{x - \sqrt{x^2 + 4z}}{2},\ t_3 = \frac{y + \sqrt{y^2 + 4z}}{2},\ t_4 = \frac{y - \sqrt{y^2 + 4z}}{2}.$$

Let $x = 2\sqrt{z} \sinh u$, so that $\sqrt{x^2 + 4z} = 2\sqrt{z} \cosh u$, and

$$t_1 = \sqrt{z}e^u \quad \text{and} \quad t_2 = -\sqrt{z}e^{-u}.$$

Let $y = 2\sqrt{z} \sinh v$, so that $\sqrt{y^2 + 4z} = 2\sqrt{z} \cosh v$, and

$$t_3 = \sqrt{z}e^v \quad \text{and} \quad t_4 = -\sqrt{z}e^{-v}.$$

Then

$$\begin{aligned}(x - y)\ell_n(x, y, z) &= z^{\frac{n}{2}}[e^{nu} + (-1)^n e^{-nu}] - z^{\frac{n}{2}}[e^{nv} + (-1)^n e^{-nv}] \\ &= \begin{cases} 2z^{\frac{n}{2}}(\sinh nu - \sinh nv) & \text{for odd } n, \\ 2z^{\frac{n}{2}}(\cosh nu - \cosh nv) & \text{for even } n. \end{cases}\end{aligned}$$

Dividing by $x - y = 2\sqrt{z}(\sinh u - \sinh v)$, we find

$$\ell_n(x, y, z) = \begin{cases} z^{\frac{n-1}{2}} \dfrac{\sinh nu - \sinh nv}{\sinh u - \sinh v} & \text{for odd } n, \\[2ex] z^{\frac{n-1}{2}} \dfrac{\cosh nu - \cosh nv}{\sinh u - \sinh v} & \text{for even } n. \end{cases}$$

Now suppose n is odd. Then $\ell_n(x, y, z) = 0$ when

$$\sinh nu = \sinh nv \text{ and } \sinh u \neq \sinh v;$$

i.e., when $nu = nv + 2k\pi i$ and k is not an integral multiple of n. Thus,

$$\ell_n(x, y, z) = 0 \text{ when } u = v + 2k\pi i/n \text{ for } k = 1, 2, \ldots, n - 1.$$

For even n we similarly reach the same result. Substitution for u and v now completes the proof.

2. GENERALIZED CYCLOTOMIC POLYNOMIALS

Following the treatment of cyclotomic polynomials in Nagell [9, p. 158], for $n \geq 2$ let $p_1, p_2, \ldots, p_r$ be the distinct prime factors of n; let

$$\Pi_0 = \ell_n,$$

and for $1 \leq k \leq r$, let

$$\Pi_k = \Pi \ell_{n/p_{i_1}p_{i_2}\cdots p_{i_k}}$$

the product extending over all the k indices i_j which satisfy the conditions

$$1 \leq i_1 < i_2 < \cdots < i_k \leq r.$$

Lemma 2: Let $C_1(x, y, z) = 1$, and for $n \geq 2$, let

$$C_n(x, y, z) = \frac{\Pi_0 \Pi_2 \cdots}{\Pi_1 \Pi_3 \cdots}. \tag{3}$$

The number of factors ℓ_q in the numerator equals the number of factors ℓ_q in the denominator.

Proof: First consider the number of ℓ_q's in the numerator: for $0 \leq j \leq [r/2]$ there are $\binom{r}{2j}$ of the ℓ_q's in Π_{2j}, so that the number we seek is

$$\sum_{j=0}^{[r/2]} \binom{r}{2j}.$$

Similarly, we count $\displaystyle\sum_{j=0}^{[(r-1)/2]} \binom{r}{2j+1}$ factors ℓ_q in the denominator. That these two sums are equal for any $r \geq 1$ follows from the identity

$$\sum_{k=0}^{r} (-1)^k \binom{r}{k} = (1 - 1)^r = 0.$$

Let us recall now some facts about cyclotomic polynomials (e.g., [9]): In case $\ell_n = x^n - 1$, the quotient C_n in (3) defines, for $n \geq 2$, the nth cyclotomic polynomial, which is irreducible over the ring of integers. (The first cyclotomic polynomial is defined to be $x - 1$). Thus, for $n \geq 1$, the roots of the nth cyclotomic polynomial are the primitive nth roots of unity: $e^{2k\pi i/n}$ where $(k, n) = 1$. Writing $\phi(n)$ for Euler's phi-function, the nth cyclotomic polynomial therefore has degree $\phi(n)$.

Referring to (2), let us call the root

$$2\sqrt{z}\sinh(\sinh^{-1} y/2\sqrt{z} + 2k\pi i/n)$$

a *primitive nth root* of $\ell_n(x, y, z)$ if $(k, n) = 1$.

Theorem 2: For $n \geq 2$, the quotient $C_n(x, y, z)$ in (3) is a polynomial with integer coefficients, having degree $\phi(n)$ in x. Moreover, for $n \geq 2$, $C_n(x, 1, 0)$ is the nth cyclotomic polynomial.

Proof: Suppose $n \geq 2$. By Lemma 2, if the quotient in (3) is formed with the polynomials $(x - 1)\ell_n(x, 1, 0)$ in the products Π_k instead of $\ell_n(x, 1, 0)$, then the result is $C_n(x, 1, 0)$. But

$$(x - 1)\ell_n(x, 1, 0) = x^n - 1,$$

so that $C_n(x, 1, 0)$ is the nth cyclotomic polynomial, which has degree $\phi(n)$ in x.

It remains to be proved that $C_n(x, y, z)$ is a polynomial for $n \geq 2$; i.e., that the polynomial $D = \Pi_1\Pi_3 \ldots$ divides the polynomial $N = \Pi_0\Pi_2 \ldots$ over the ring of integers. Since this is the case for $(x, 1, 0)$, each linear factor $x - r$ of D is a factor of N and must occur at least as many times in N as in D. But each such r is an nth root of unity, $r = e^{2k\pi i/n}$ for some k and n. So in the general case (x, y, z), each linear factor $x - 2\sqrt{z}\,\sinh(\sinh^{-1}y/2\sqrt{z} + 2k\pi i/n)$ of D occurs at least as many times in N as in D. Thus, D divides N. Since all the coefficients of N and D have only integer coefficients, the same must be true of the quotient $C_n(x, y, z)$, by the division algorithm for polynomials in x over the ring $I[y, z]$ of bivariate polynomials with integer coefficients.

Theorem 3: For $n \geq 2$,

$$C_n(x, y, z) = \prod_{\substack{(k,n)=1 \\ 0 \leq k \leq n}} [x - 2\sqrt{z}\,\sinh(\sinh^{-1}y/2\sqrt{z} + 2k\pi i/n)].$$

Proof: This is an obvious consequence of the one-to-one correspondence between roots of $C_n(x, y, z)$ and roots of the nth cyclotomic polynomial

$$C_n(x, 1, 0) = \prod_{\substack{(k,n)=1 \\ 0 \leq k \leq n}} (x - e^{2k\pi i/n}).$$

Theorem 4: For $n \geq 1$,

$$\ell_n(x, y, z) = \prod_{d|n} C_d(x, y, z).$$

Proof: First, $\ell_1(x, y, z) = C_1(x, y, z) = 1$. Now suppose $n \geq 2$. Then

$$C_d(x, y, z) = (x - r_1) \ldots (x - r_{\phi(d)}),$$

where the r_i's range through the roots $2\sqrt{z}\,\sinh(\sinh^{-1}y/2\sqrt{z} + 2k\pi i/n)$ of $\ell_d(x, y, z)$ for which $(k, d) = 1$. Each root of $\ell_n(x, y, z)$ is a primitive dth root of one and only one $C_d(x, y, z)$ where $d|n$. Thus each linear factor of $\ell_n(x, y, z)$ occurs in one and only one $C_d(x, y, z)$.

Lemma 5: For $n \geq 1$, the polynomial $C_n(x, y, 0)$ is irreducible over the ring of integers.

Proof: The statement is clearly true for $n = 1$. For $n \geq 2$, suppose

$$C_n(x, y, 0) = d(x, y)q(x, y).$$

Then

$$C_n(x, 1, 0) = d(x, 1)q(x, 1).$$

Since the cyclotomic polynomial $C_n(x, 1, 0)$ is irreducible, one of the polynomials $d(x, 1)$ and $q(x, 1)$ must be the constant 1 polynomial. Without any loss, we may suppose this one to be $d(x, 1)$ and thus have

$$d(x, y) = 1 + (y - 1)e(x, y)$$

for some polynomial $e(x, y)$. Then

$$C_n(x, y, 0) = q(x, y) + (y - 1)e(x, y)q(x, y).$$

Now $q(x, y)$ includes the term $x^{\phi(n)}$, which cannot appear in

$$(y - 1)e(x, y)q(x, y).$$

Therefore, $e(x, y) = 0$, so that $d(x, y) = 1$.

Theorem 5: For $n \geq 1$, the polynomial $C_n(x, y, z)$ is irreducible over the ring of integers.

Proof: Suppose

$$C_n(x, y, z) = d(x, y, z)q(x, y, z).$$

Then

$$C_n(x, y, 0) = d(x, y, 0)q(x, y, 0).$$

By Lemma 5, one of the polynomials $d(x, y, 0)$ and $q(x, y, 0)$ is the constant 1 polynomial. Consequently, as in the proof of Lemma 5, we have

$$d(x, y, z) = 1 + ze(x, y, z)$$

for some polynomial $e(x, y, z)$. Then

$$C_n(x, y, z) = q(x, y, z) + ze(x, y, z)q(x, y, z).$$

Now $q(x, y, z)$ includes the term $x^{\phi(n)}$, which cannot appear in

$$ze(x, y, z)q(x, y, z).$$

Therefore, $e(x, y, z) = 0$, so that $d(x, y, z) = 1$.

TABLE 1

Generalized Cyclotomic Polynomials $C_n = C_n(x, y, z)$

$$C_1 = 1$$

$$C_2 = x + y$$

$$C_3 = x^2 + xy + y^2 + 3z$$

$$C_4 = x^2 + y^2 + 4z$$

$$C_5 = x^4 + x^3y + x^2y^2 + xy^3 + y^4 + 5z(x^2 + xy + y^2) + 5z^2$$

$$C_6 = x^2 - xy + y^2 + 3z$$

$$C_8 = x^4 + y^4 + 4z(x^2 + y^2) + 4z^2$$

$$C_9 = x^6 + x^3y^3 + y^6 + 3z(2x^4 + x^3y + xy^3 + 2y^4) + 9z^2(x^2 + xy + y^2) + 3z^3$$

$$C_{10} = (x^5 + y^5)/(x + y) + 5z(x^3 + y^3)/(x + y) + 5z^2$$

$$C_{12} = x^4 - x^2y^2 + y^4 + 2z(x^2 + y^2) + z^2$$

Abbreviating $C_n(x, y, 0)$ as c_n, we note that

$$C_3 = c_3 + 3z,\ C_4 = c_4 + 4z,\ C_6 = c_6 + 3z,\ C_8 = c_8 + 4zc_4 + 4z^2,$$

$$C_{10} = c_{10} + 5zc_6 + 5z^2,\ C_{12} = c_{12} + 2zc_4 + z^2,$$

and

$$C_9 = c_9 + 3z(c_5 + c_{12}) + 9z^2c_3 + 3z^3.$$

One wonders if all the coefficients of powers of z are linear combinations of c_i's.

3. THE CASE $z = 0$: FIBONACCI CYCLOTOMIC POLYNOMIALS

Here we will determine the irreducible factors of the generalized Fibonacci polynomials. In Section 1, the (not generalized) irreducible factors were named the Fibonacci cyclotomic polynomials and denoted $\mathcal{F}_n(x)$. Here, however, we shall deal with the natural generalization: the *generalized Fibonacci cyclotomic polynomials*, denoted $\mathcal{F}_n(x, y)$. Theorem 6 will show that

$$\mathcal{F}_n(x, y) = C_n\left(\frac{x + \sqrt{x^2 + 4y}}{2}, \frac{x - \sqrt{x^2 + 4y}}{2}, 0\right) \text{ for } n \geq 1,$$

and Corollary 7 will show that the $\mathcal{F}_n(x)$'s can be expressed as linear combinations of generalized (unmodified) Lucas polynomials.

Theorem 6: For $n \geq 1$, let $F_n(x, y)$ be the nth generalized Fibonacci polynomial. Then

$$F_n(x, y) = \prod_{d|n} C_d\left(\frac{x + \sqrt{x^2 + 4y}}{2}, \frac{x - \sqrt{x^2 + 4y}}{2}, 0\right).$$

Moreover, the polynomials $C_d\left(\frac{x + \sqrt{x^2 + 4y}}{2}, \frac{x - \sqrt{x^2 + 4y}}{2}, 0\right)$, as polynomials in x and y, are irreducible over the ring of integers.

Proof: Write $s = \frac{x + \sqrt{x^2 + 4y}}{2}$ and $t = \frac{x - \sqrt{x^2 + 4y}}{2}$. By (1) and Theorem 4,

$$F_n(x, y) = \ell_n(s, t, 0) = \prod_{d|n} C_d(s, t, 0).$$

To see that the C_d's are irreducible as polynomials in x and y, suppose

$$C_d(s, t, 0) = p(x, y)q(x, y).$$

Then, since $x = s + t$ and $y = -st$, we have $C_d(s, t, 0)$ written as a product of two polynomials each in s and t. By Lemma 5, one of these polynomials is a constant polynomial, namely 1, since C_d is monic. Thus, either $p(x, y) = 1$ or $q(x, y) = 1$, as desired.

Theorem 7: For $k \geq 1$, let $L_k(x, y)$ be the kth generalized (unmodified) Lucas polynomial. For $n \geq 3$, the nth generalized Fibonacci cyclotomic polynomial is given by

$$\mathcal{F}_n(x, y) = \sum_{j=0}^{\phi(n)/2} \delta_j y^{\frac{\phi(n)}{2} - j} L_{2j}(x, y),$$

where $\delta_{\phi(n)/2} = 1$ and the numers $\delta_0, \delta_1, \delta_2, \ldots, \delta_{\frac{\phi(n)}{2}-1}$ are integers.

Proof: Suppose $n \geq 3$. With s and t as in the proof of Theorem 6,

$$\mathcal{F}_n(x, y) = C_n(s, t, 0) = t^{\phi(n)} C_n(s/t, 1, 0),$$

where

$$C_n(u, 1, 0) = u^{\phi(n)} + a_{\phi(n)-1} u^{\phi(n)-1} + \cdots + a_1 u + 1$$

is the nth cyclotomic polynomial. Thus, $C_n(s, t, 0)$ has the form

$$s^{\phi(n)} + a_{\phi(n)-1} s^{\phi(n)-1} t + \cdots + a_1 s t^{\phi(n)-1} + t^{\phi(n)}.$$

Since $C_n(s, t, 0)$ is symmetric in s and t, this polynomial is expressible as

$$s^{\phi(n)} + t^{\phi(n)} + a_{\phi(n)-1} st\left(s^{\phi(n)-2} + t^{\phi(n)-2}\right) + \cdots + a_{\frac{\phi(n)}{2}} (st)^{\frac{\phi(n)}{2}}.$$

Recalling $st = -y$ and the Binet formula $L_k(x, y) = s^k + t^k$ [in particular, $L_0(x, y) = 2$], we conclude that

$$\mathcal{F}_n(x, y) = L_{\phi(n)} - a_{\phi(n)-1} y L_{\phi(n)-2} + \cdots + \frac{(-1)^{\frac{\phi(n)}{2}}}{2} a_{\frac{\phi(n)}{2}} y^{\frac{\phi(n)}{2}} L_0,$$

as desired.

Corollary 7: Only for the purpose of facilitating the statement of this corolary, suppose $L_0(x, y) = 1$ (instead of 2). Then for $n \geq 1$, the nth Fibonacci cyclotomic polynomial $\mathcal{F}_n(x)$ is an integral linear combination of Lucas polynomials $L_n(x)$.

Proof: The proposition is easily verified for $n = 0, 1, 2$. For $n \geq 3$, put $y = 1$ in Theorem 7.

To illustrate Corollary 7, we write out, in Table 2, several Fibonacci cyclotomic polynomials $\mathcal{F}_n = \mathcal{F}_n(x, 1)$ in terms of the Lucas polynomials $L_n = L_n(x, 1)$. Recall that the $\mathcal{F}_n$'s are the irreducible divisors of the Fibonacci polynomials, in accord with the identity

$$F_n = \prod_{d|n} \mathcal{F}_d.$$

TABLE 2

Fibonacci Cyclotomic Polynomials

degree 0: $\mathcal{F}_1 = 1$

degree 1: $\mathcal{F}_2 = x = L_1$

degree 2: $\mathcal{F}_3 = x^2 + 1 = L_2 - 1$

$\mathcal{F}_4 = x^2 + 2 = L_2$

$\mathcal{F}_6 = x^2 + 3 = L_2 + 1$

degree 4: $\mathcal{F}_5 = x^4 + 3x^2 + 1 = L_4 - L_2 + 1$

$\mathcal{F}_8 = x^4 + 4x^2 + 2 = L_4$

$\mathcal{F}_{10} = x^4 + 5x^2 + 5 = L_4 + L_2 + 1$

$\mathcal{F}_{12} = x^4 + 4x^2 + 1 = L_4 - 1$

TABLE 2 (continued)

degree 6:
$$\mathcal{F}_7 = x^6 + 5x^4 + 6x^2 + 1 = L_6 - L_4 + L_2 - 1$$
$$\mathcal{F}_9 = x^6 + 6x^4 + 9x^2 + 1 = L_6 - 1$$
$$\mathcal{F}_{14} = x^6 + 7x^4 + 14x^2 + 7 = L_6 + L_4 + L_2 + 1$$
$$\mathcal{F}_{18} = x^6 + 6x^4 + 9x^2 + 4 = L_6 + 1$$

degree 8:
$$\mathcal{F}_{15} = x^8 + 9x^6 + 26x^4 + 24x^2 + 1 = L_8 + L_6 - L_2 - 1$$
$$\mathcal{F}_{16} = x^8 + 8x^6 + 20x^4 + 16x^2 + 2 = L_8$$
$$\mathcal{F}_{20} = x^8 + 8x^6 + 19x^4 + 12x^2 + 1 = L_8 - L_4 + 1$$
$$\mathcal{F}_{24} = x^8 + 8x^6 + 20x^4 + 16x^2 + 1 = L_8 - 1$$
$$\mathcal{F}_{30} = x^8 + 7x^6 + 14x^4 + 8x^2 + 1 = L_8 - L_6 + L_2 - 1$$

degree > 8:
$$\mathcal{F}_{11} = L_{10} - L_8 + L_6 - L_4 + L_2 - 1$$
$$\mathcal{F}_{32} = L_{16}$$
$$\mathcal{F}_{33} = L_{20} + L_{18} - L_{14} - L_{12} + L_8 + L_6 - L_2 - 1$$
$$\mathcal{F}_{36} = L_{12} - 1$$
$$\mathcal{F}_{40} = L_{16} - L_8 + 1$$
$$\mathcal{F}_{42} = L_{12} - L_{10} + L_6 - L_4 + 1$$
$$\mathcal{F}_{45} = L_{24} + L_{18} - L_6 - 1$$
$$\mathcal{F}_{48} = L_{16} - 1$$
$$\mathcal{F}_{50} = L_{20} + L_{10} + 1$$
$$\mathcal{F}_{105} = L_{48} - L_{46} + L_{44} + L_{38} - L_{36} + 2L_{34} - L_{32} + L_{30} + L_{24} - L_{22} + L_{20} - L_{18} + L_{16} - L_{14} - L_8 - L_4 - 1$$

Note in particular the coefficient of L_{34} in the polynomial $\mathcal{F}_{105}$.

Two reminders (e.g., [9]) about the cyclotomic polynomials $C_n(u, 1, 0) = \Phi_n(u)$ which are helpful in computing $\mathcal{F}_n$'s are the following:

(i) If p is a prime and $p \nmid n$, then $\Phi_{np}(u) = \Phi_n(u^p)/\Phi_n(u)$;

(ii) If p is a prime and $p \mid n$, then $\Phi_{np}(u) = \Phi_n(u^p)$.

As an example, we compute $\mathcal{F}_{45}$ as follows:

$$\Phi_{45}(u) = \Phi_{15}(u^3) = \Phi_3(u^{15})/\Phi_3(u^3) = \frac{u^{30} + u^{15} + 1}{u^6 + u^3 + 1}$$
$$= u^{24} - u^{21} + u^{15} - u^{12} + u^9 - u^3 + 1,$$

so that

$$\begin{aligned}\mathcal{F}_{45}(x, y) &= C_{45}(s, t, 0)\\ &= s^{24} - s^{21}t^3 + s^{15}t^9 - s^{12}t^{12} + s^9t^{15} - s^3t^{21} + t^{24}\\ &= s^{24} + t^{24} - (st)^3(s^{18} + t^{18}) + (st)^9(s^6 + t^6) - (st)^{12}\end{aligned}$$

$$= L_{24} + y^3 L_{18} - y^9 L_6 - y^{12} \quad \text{(Theorem 7)},$$
$$\mathcal{F}_{45}(x,\ 1) = L_{24} + L_{18} - L_6 - 1 \quad \text{(Corollary 7)}.$$

Since for highly composite values of n the cyclotomic polynomials tend to be complicated ([1], [3], [4], [11], [12]), the same is true for the corresponding Fibonacci cyclotomic polynomials.

In Theorem 12 of [6], Hoggatt and Long find an upper bound for the number $N(m)$ of polynomials of degree $2m$ that divide some Fibonacci polynomial. If we restrict $N(m)$ to irreducible polynomials, then $N(m)$ is the number of solutions n to the equation $\phi(n) = 2m$. For example, $N(720) = 72$. That is, there are 72 distinct Fibonacci cyclotomic polynomials $\mathcal{F}_n$ having degree 1440. See [10].

Still restricting $N(m)$ to irreducible polynomials, we ask if $N(m) = 0$ for any m. The answer is yes. C. L. Klee proved in [8] that $\phi(n) = 2m$ has no solution n if m has no divisor $d > 1$ for which $2d+1$ is a prime. For example, no $\mathcal{F}_n$ has degree 14.

4. THE CASE $y = 0$: LUCAS CYCLOTOMIC POLYNOMIALS

Our main objective in this section is to determine the irreducible factors of the generalized modified Lucas polynomials $\ell_n(x,\ 0,\ z)$. First, however, we wish to justify the names *Lucas cyclotomic polynomials* and *generalized Lucas cyclotomic polynomials* for the sequences

$$C_n(x,\ 0,\ 1) \quad \text{and} \quad C_n(x,\ 0,\ z),$$

since these sequences are determined by (3) from the generalized *modified* Lucas sequence $\ell_n(x,\ 0,\ z)$ and not the generalized Lucas sequence $L_n(x,\ z)$. The justification is this: that, by Theorem 1, the quotient (3) defines polynomials analogous to cyclotomic polynomials in the former case, but does not generally define polynomials at all if the L_n's are substituted for the ℓ_n's. (Nevertheless, the irreducible factors of the L_n's will be easily determined otherwise in Section 5.)

In Section 1, the (not generalized) Lucas analogue of the Fibonacci cyclotomic polynomials were named Lucas cyclotomic polynomials and denoted by $\mathcal{L}_n(x)$. Here however, we shall deal with the natural generalization, the generalized Lucas cyclotomic polynomials, denoted $\mathcal{L}_n(x,\ z)$ and defined by

$$\mathcal{L}_n(x,\ z) = C_n(x,\ 0,\ z).$$

By Theorem 3 and the identity $\sinh iu = i \sin u$, the roots of $\mathcal{L}_n(x,\ z)$ are

$$2i\sqrt{z}\ \sin 2k\pi/n,\ (k,\ n) = 1,\ 1 \le k \le n-1.$$

The roots of $F_n(x,\ z)$ are $2i\sqrt{z}\ \cos k\pi/n$ for $1 \le k \le n-1$, as proved in [5] and [6], and consequently, the roots of $\mathcal{F}_n(x,\ z)$ are

$$2i\sqrt{z}\ \cos k\pi/n,\ (k,\ n) = 1,\ 1 \le k \le n-1.$$

In order to reconcile roots of the $\mathcal{L}_n(x,\ z)$'s with those of the $\mathcal{F}_n(x,\ z)$'s let

$$Q_n = \{k : (k,\ n) = 1 \ \text{ and } \ 1 \le k \le n-1\}.$$

for $k \in Q_n$, we have

$$\sin 2k\pi/n = \cos(n - 4k)\pi/2n.$$

As k ranges through the set Q_n, it is natural to expect the numbers $n - 4k$ to range through residue sets modulo various divisors or multiples of n. Such expectations are fulfilled in the next theorem.

Theorem 8: Except for $\mathcal{L}_1(x, z) = 1$ and $\mathcal{L}_4(x, z) = x^2 + 4z$, the nth generalized Lucas cyclotomic polynomial $\mathcal{L}_n(x, z)$ can be expressed in terms of the generalized Fibonacci cyclotomic polynomials as follows:

$$\mathcal{L}_n(x, z) = \begin{cases} \mathcal{F}_{2n}(x, z) & \text{for odd } n,\ n \neq 1, \\ \mathcal{F}_n(x, z) & \text{for } n = 2q,\ q \text{ odd}, \\ \mathcal{F}_q^2(x, z) & \text{for } n = 4q,\ q \text{ odd},\ q \neq 1, \\ \mathcal{F}_{2^t q}^2(x, z) & \text{for } n = 2^{t+1}q,\ q \text{ odd},\ t \geq 2. \end{cases}$$

Proof:

Case 1. Suppose n is odd and $n \neq 1$. Then

$$\cos\frac{(n - 4k)\pi}{2n} = \begin{cases} \cos\dfrac{|n - 4k|\pi}{2n} & \text{for } 4k < 3n, \\ \cos\dfrac{(5n - 4k)\pi}{2n} & \text{for } 4k > 3n. \end{cases}$$

Let

$$A = \{|n - 4k| : k \in Q_n \text{ and } 4k < 3n\},$$
$$B = \{5n - 4k : k \in Q_n \text{ and } 4k > 3n\},$$

and

$$Q = A \cup B.$$

It suffices to show that $Q = Q_{2n}$ and that each element of Q_{2n} appears only once in forming the set Q. This will be shown in four steps:

(i) $A \cap B$ is empty;
(ii) Q consists of $\phi(2n)$ elements;
(iii) If $j \in Q$, then $1 \leq j \leq 2n - 1$;
(iv) If $j \in Q$, then $(j, 2n) = 1$.

To verify (i), suppose $n - 4k_1 = 5n - 4k_2$ where $4k_1 < 3n$ and $4k_2 > 3n$. Then $k_2 - k_1 = n$, contrary to the inequalities

$$1 \leq k_1 \leq n - 1 \quad \text{and} \quad 1 \leq k_2 \leq n - 1.$$

If $|n - 4k_1| = 4k_1 - n = 5n - 4k_2$, then $2(k_1 + k_2) = 3n$, contrary to our assumption that n is odd.

For (ii), we know from (i) that distinct k's in Q_n provide distinct elements in Q. Furthermore, every element k in Q_n does yield an element of A or B, since $4k = 3n$ is impossible for odd n. Thus, Q consists of the same number of elements as Q_n, which is $\phi(n)$. Since n is odd, we have $\phi(n) = \phi(2n)$.

To verify (iii), first suppose $4k < 3n$. If $n - 4k \geq 0$, then $1 \leq n - 4k$ since n is an odd positive integer and, clearly, $n - 4k \leq 2n - 1$; if $n - 4k < 0$, then, similarly, $1 \leq 4k - n$, and $4k - n \leq 2n - 1$ since $4k < 3n$. Now suppose $4k > 3n$. Then $5n - 4k \leq 2n - 1$, and also $1 \leq 5n - 4k$, since $k < n$.

For (iv), if $d|(n-4k)|$ and $d|2n$, then d must be odd since $n-4k$ is odd. Consequently, $d|n$. But then $d|4k$, so that $d|k$. Since $(k, n) = 1$, we conclude that $(n-4k, 2n) = 1$. The same clearly holds for $4k-n$ and $5n-4k$.

Case 2. Suppose $n = 2q$, q odd. Then

$$\cos\frac{(n-4k)\pi}{2n} = \begin{cases} \cos\dfrac{|q-2k|\pi}{n} & \text{for } 2k < 3q, \\ \cos\dfrac{(5q-2k)\pi}{n} & \text{for } 2k > 3q. \end{cases}$$

Here, the numbers $|q-2k|$ and $5q-2k$, as stipulated, range through the set Q_n as k ranges through the set Q_n. The proof is so similar to that in Case 1 that we omit it here.

Case 3. Suppose $n = 4q$, q odd, $q \neq 1$. Let

$$A = \{k \in Q_n : k < q\},\ B = \{k \in Q_n : q < k < 2q\},$$
$$C = \{k \in Q_n : 2q < k < 3q\},\ D = \{k \in Q_n : 3q < k\}.$$

Each k in Q_n in odd, so that $(q-k)/2$ is an integer, and

$$\cos\frac{(n-4k)\pi}{2n} = \begin{cases} \cos\dfrac{|(q-k)/2|\pi}{q} & \text{for } k \in A \cup B, \\ \cos\dfrac{[(5q-k)/2]\pi}{q} & \text{for } k \in C \cup D. \end{cases}$$

We first claim that as k ranges through the set $A \cup C$, the numbers $|(q-k)/2|$ and $(5q-k)/2$, as stipulated, range through the set Q_q. This claim is verified as in the four steps in Case 1. Starting with

$$A^* = \{|(q-k)/2| : k \in A\} \quad\text{and}\quad C^* = \{(5q-k)/2 : k \in C\},$$

only step (ii) calls for anything new: To see that $A^* \cup C^*$ consists of $\phi(q)$ elements [granted from step (i) that distinct k's lead to distinct elements in $A \cup B \cup C \cup D$], we note that the number of k's in Q_n is

$$\phi(4q) = \phi(4)\phi(q) = 2\phi(q),$$

and precisely half of these lie in $A^* \cup C^*$ since, as is easily checked, the sets A, B, C, D are in one-to-one correspondence with one another:

$$\begin{aligned} A \to B &: k \to 2q - k, \\ A \to C &: k \to 2q + k, \\ C \to D &: k \to 6q - k. \end{aligned}$$

Thus, the roots of $\mathcal{L}_n(x, z)$ found for $k \in A \cup C$ are the roots of $\mathcal{F}_q(x, z)$. That the same is true for $k \in B \cup D$ will now be proved. Since

$$B = \{2q - k : k \in A\},$$

we have

$$\left\{\cos\frac{(n-4k)\pi}{2n} : k \in B\right\} = \left\{\cos\frac{|(q-k)/2|\pi}{q} : k \in A\right\}.$$

Since $D = \{6q - k : k \in C\}$, we have

$$\left\{\cos\frac{(n-4k)\pi}{2n} : k \in D\right\} = \left\{\cos\frac{[(5q-k)/2]\pi}{q} : k \in C\right\}.$$

Thus the roots of $\mathcal{L}_n(x, z)$ for $k \in B \cup D$ are the roots of $\mathcal{F}_q(x, z)$. We conclude that $\mathcal{L}_n(x, z) = \mathcal{F}_q^2(x, z)$.

Case 4. Suppose $n = 2^{t+1}q$, q odd, $t \geq 2$. Define sets A, B, C, D as in Case 3, and have the following one-to-one correspondences:

$$\begin{aligned} A \to B &: k \to 2^t q - k, \\ A \to C &: k \to 2^t q + k, \\ C \to D &: k \to 3 \cdot 2^t q - k. \end{aligned}$$

Now

$$\cos\frac{(n-4k)\pi}{2n} = \begin{cases} \cos\dfrac{|2^{t-1}q - k|\pi}{2^t q} & \text{for } k \in A \cup B, \\[2ex] \cos\dfrac{(5 \cdot 2^{t-1}q - k)\pi}{2^t q} & \text{for } k \in C \cup D. \end{cases}$$

We claim that as k ranges through the set $A \cup C$, the numbers $|2^{t-1}q - k|$ and $(5 \cdot 2^{t-1}q - k)$, as stipulated, range through the set $Q_{2^t q}$. The four steps in Case 3 easily verify this claim. We omit the verification, except to note that for step (ii) we have $\phi(2^{t+1}q) = 2\phi(2^t q)$, so that $\phi(2^t q)$ roots are found for $k \in A \cup C$.

As in Case 3, we have

$$\left\{\cos\frac{(n-4k)\pi}{2n} : k \in B \cup D\right\} = \left\{\cos\frac{(n-4k)\pi}{2n} : k \in A \cup C\right\}.$$

Therefore, $\mathcal{L}_n(x, z) = \mathcal{F}_{2^t q}^2$, and Theorem 8 is proved.

Theorem 8 and Theorem 4 enable us to factor the polynomials $\ell_n(x, 0, z)$ completely in terms of irreducible factors. For example,

$$\begin{aligned} \ell_{60}(x, 0, z) &= \prod_{d|60} C_d(x, 0, z) \\ &= \prod_{d|60} \mathcal{L}_d(x, z) \\ &= \mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\mathcal{L}_4\mathcal{L}_5\mathcal{L}_6\mathcal{L}_{10}\mathcal{L}_{12}\mathcal{L}_{15}\mathcal{L}_{20}\mathcal{L}_{30}\mathcal{L}_{60} \\ &= x(x^2 + 4z)(\mathcal{F}_3\mathcal{F}_5\mathcal{F}_6\mathcal{F}_{10}\mathcal{F}_{15}\mathcal{F}_{30})^2. \end{aligned}$$

Recalling that $F_{30} = \mathcal{F}_2\mathcal{F}_3\mathcal{F}_5\mathcal{F}_6\mathcal{F}_{10}\mathcal{F}_{15}\mathcal{F}_{30}$, that $x\ell_{60}(x, 0, z) = L_{60} - 2z^{30}$, and that $x^2 + 4z$ is the discriminant $D(x, z)$ of $t^2 - xt - z$, we rewrite L_{60} as follows:

$$L_{60}(x, z) = D(x, z)F_{30}^2(x, z) + 2z^{30}.$$

Putting $x = z = 1$, we find an identity $L_{60} = 5F_{30}^2 + 2$ involving the thirtieth Fibonacci number and the sixtieth Lucas number. These considerations lead to the following theorems and corollary.

Theorem 9a: Suppose $m = 2^t q$, q odd, $t \geq 2$. Then

$$L_{2m}(x, z) = (x^2 + 4z)F_m^2(x, z) + 2z^m. \tag{4}$$

Proof:
$$\begin{aligned}\ell_{2m} &= \mathcal{L}_1\mathcal{L}_2\mathcal{L}_4 \ldots \mathcal{L}_{2^{t+1}}\mathcal{L}_q\mathcal{L}_{2q}\mathcal{L}_{4q} \ldots \mathcal{L}_{2^{t+1}q}\\ &= x(x^2 + 4z)\mathcal{F}_4^2\mathcal{F}_8^2 \ldots \mathcal{F}_{2^t}^2\mathcal{F}_{2q}^2\mathcal{F}_q^2\mathcal{F}_{4q}^2\mathcal{F}_{8q}^2 \ldots \mathcal{F}_{2^tq}^2\\ &= x(x^2 + 4z)F_m^2/x^2,\end{aligned}$$

and (4) follows immediately.

Theorem 9b: If m is odd, then

$$L_{2m}(x, z) - 2z^m = L_m^2(x, z). \tag{5}$$

Proof: The proof of this known identity is so similar to that of Theorem 9a that we omit it here.

Corollary 9: For $k > 0$, let F_k and L_k be the kth Fibonacci and Lucas numbers. If $m = 2^t q$, q odd, $t \geq 2$, then

$$L_{2m} = 5F_m^2 + 2.$$

If m is odd, then

$$L_{2m} = L_m^2 + 2.$$

Proof: Put $x = z = 1$ in (4) and (5).

5. THE IRREDUCIBLE FACTORS OF THE LUCAS POLYNOMIALS

Hoggatt and Bicknell prove in [5] that for $n \geq 1$ the roots of the nth Lucas polynomial $L_n(x, 1)$ are

$$2i \cos\frac{(2k + 1)\pi}{2n}, \quad k = 0, 1, \ldots, n - 1.$$

The methods of Section 4 could be used to compare these roots with those of the Fibonacci cyclotomic polynomials. However, we choose a different way, which depends on the well-known identity $F_{2n} = L_nF_n$.

Theorem 10: For $n \geq 1$, write $n = 2^t q$, where $t \geq 0$ and q is odd. The nth generalized Lucas polynomial $L_n(x, z)$ is a product of (irreducible) Fibonacci cyclotomic polynomials:

$$L_n(x, z) = \prod_{d|q}\mathcal{F}_{2^{t+1}d}(x, z).$$

Proof:

$$L_n = \frac{F_{2n}}{F_n} = \frac{\prod_{d|2n}\mathcal{F}_d}{\prod_{d|n}\mathcal{F}_d} = \prod_{\substack{d|2n\\ d\nmid n}}\mathcal{F}_d.$$

Now

$$\{d : d|2n \text{ and } d\nmid n\} = \{2^{\,+1}d : d|n \text{ and } d \text{ is odd}\},$$

so that the conditions $d|2n$, $d\nmid n$ are replaceable by the condition $2^{t+1}d|2n$, i.e., $d|q$.

Example:

$$L_{60} = \frac{\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3\mathcal{F}_4\mathcal{F}_5\mathcal{F}_6\mathcal{F}_8\mathcal{F}_{10}\mathcal{F}_{12}\mathcal{F}_{15}\mathcal{F}_{20}\mathcal{F}_{24}\mathcal{F}_{30}\mathcal{F}_{40}\mathcal{F}_{60}\mathcal{F}_{120}}{\mathcal{F}_1\mathcal{F}_2\mathcal{F}_3\mathcal{F}_4\mathcal{F}_5\mathcal{F}_6\mathcal{F}_{10}\mathcal{F}_{12}\mathcal{F}_{15}\mathcal{F}_{20}\mathcal{F}_{30}\mathcal{F}_{60}}$$

$$= \mathcal{F}_8\mathcal{F}_{24}\mathcal{F}_{40}\mathcal{F}_{120}.$$

Corollary 10: For even $n \geq 2$, $L_n(x, z)$ is irreducible if and only if $n = 2^k$ for some $k \geq 1$.

Proof: Suppose $n = 2^k$ for some $k \geq 1$. Then by Theorem 10, we have $L_n = \mathcal{F}_{2n}$, which is irreducible by Theorem 6. If n is even but not a power of 2, then by Theorem 10, $\mathcal{F}_{2n}$ is a proper divisor of $L_n(x, z)$.

In [2], Bergum and Hoggatt prove Corollary 10 using Eisenstein's Criterion.

We conclude this section by noting that the divisibility properties that are already established for the polynomials F_n, L_n, and ℓ_n in terms of the irreducible polynomials $\mathcal{F}_n$ now carry over to divisibility properties of Chebyshev polynomials of the first and second kinds.

It is well known that the nth Chebyshev polynomial of the first kind is

$$T_n(x) = \frac{1}{2}L_n(2x, -1), \; n = 0, 1, \ldots .$$

Accordingly, the factorization of $T_n(x)$ in terms of factors which are irreducible over the ring of integers is given by Theorem 10.

Let us define *modified Chebyshev polynomials of the first kind* by

$$t_n(x) = \begin{cases} \dfrac{1}{x}T_n(x) & \text{for odd } n, \\[2ex] \dfrac{1}{x}\left[T_n(x) - (-1)^{\frac{n}{2}}\right] & \text{for even } n > 0. \end{cases}$$

Then we have $t_n(x) = \frac{1}{2}\ell_n(2x, 0, -1)$, so that the divisibility properties of the t_n's are the same as those of the ℓ_n's. In particular, the irreducible factors are given by Theorem 8. Moreover, many of the results proved in [7] [e.g., concerning greatest common divisors, $(\ell_m, \ell_n) = \ell_{(m,n)}$] carry over to similar results for the modified Chebyshev polynomials.

It is well known that the nth Chebyshev polynomial of the second kind is

$$U_n(x) = F_{n+1}(2x, -1), \; n = 0, 1, \ldots .$$

Accordingly, the factorization of $U_n(x)$ in terms of irreducible factors is given by Theorem 6.

Finally, note that the roots of the Chebyshev and modified Chebyshev polynomials, and also the roots of their irreducible factors, are easily obtained from Theorem 1 and Theorem 3.

6. TRANSFORMED FIBONACCI AND LUCAS POLYNOMIALS

For any integers (or indeterminants) a, b, c, where $a \neq 0 \neq c$, let

$$U_n(x, z) = F_n(ax, bx^2 + cz^2),$$

$$V_n(x, z) = \frac{1}{2}L_n(ax, bx^2 + cz^2),$$

and $$W_n(x, z) = \ell_n(ax, 0, bx^2 + cz^2).$$

Then the quotients (3) are clearly polynomials for each of the sequences

$$U_n(x, z) \text{ and } W_n(x, z),$$

since this is true for the sequences F_n and ℓ_n. Similarly, the divisibility properties of the V_n's follow from those of the L_n's, as given in [2] and Section 5.

One of the most attractive special cases is $(a, b, c) = (2, -1, 1)$. We tabulate the first few U_n's and V_n's in this case. Then we tabulate the first few W_n's and the first few *transformed Fibonacci cyclotomic polynomials*; i.e., the quotients (3) formed from the U_n's. These, we shall show, are irreducible except for a constant multiple; hence, they are the irreducible factors not only of the U_n's, but also of the V_n's and the W_n's. After the tables, we shall return to arbitrary a, b, c satisfying $a^2 + 4b = 0$ and find roots, Binet forms, etc.

TABLE 3

Transformed Generalized Fibonacci Polynomials $U_n = F_n(2x, z^2 - x^2)$

and Transformed Generalized Lucas Polynomials $V_n = \frac{1}{2}L_n(2x, z^2 - x^2)$

n	U_n	V_n
1	1	x
2	$2x$	$x^2 + z^2$
3	$3x^2 + z^2$	$x^3 + 3xz^2$
4	$4x^3 + 4xz^2$	$x^4 + 6x^2z^2 + z^4$
5	$5x^4 + 10x^2z^2 + z^4$	$x^5 + 10x^3z^2 + 5xz^4$
6	$6x^5 + 20x^3z^2 + 6xz^4$	$x^6 + 15x^4z^2 + 15x^2z^4 + z^6$
7	$7x^6 + 35x^4z^2 + 21x^2z^4 + z^6$	$x^7 + 20x^5z^2 + 35x^3z^4 + 7xz^6$

One immediately detects Pascal's triangle lurking within Table 3. We shall soon ascertain that $zU_n + V_n = (x + z)^n$ for $n \geq 1$.

TABLE 4

Transformed Generalized Modified Lucas Polynomials

$$W_n = \ell_n(2x, 0, z^2 - x^2)$$

$$\begin{aligned}
W_1 &= 1\\
W_2 &= 2x\\
W_3 &= x^2 + 3z^2\\
W_4 &= 8xz^2\\
W_5 &= x^4 + 10x^2z^2 + 5z^4\\
W_6 &= 2x^5 + 12x^3z^2 + 18xz^4\\
W_7 &= x^6 + 21x^4z^2 + 35x^2z^4 + 7z^6\\
W_8 &= 32x^5z^2 + 64x^3z^4 + 32xz^6
\end{aligned}$$

TABLE 5

Transformed Generalized Fibonacci Cyclotomic Polynomials

$$\mathcal{U}_n = \mathcal{F}_n(2x, z^2 - x^2)$$

$$\begin{aligned}
\mathcal{U}_1 &= 1\\
\mathcal{U}_2 &= 2x\\
\mathcal{U}_3 &= 3x^2 + z^2\\
\mathcal{U}_4 &= 2x^2 + 2z^2\\
\mathcal{U}_5 &= 5x^4 + 10x^2z^2 + z^4\\
\mathcal{U}_6 &= x^2 + 3z^2\\
\mathcal{U}_8 &= 2x^4 + 12x^2z^2 + 2z^4\\
\mathcal{U}_{10} &= x^4 + 10x^2z^2 + 5z^4\\
\mathcal{U}_{12} &= x^4 + 14x^2z^2 + z^4
\end{aligned}$$

Lemma 11: Suppose n is an odd positive integer ≥ 3. Then

$$\prod_{k=1}^{\frac{n-1}{2}} \cos^2 \frac{k\pi}{n} = 2^{1-n}, \quad \prod_{k=0}^{\frac{n-3}{2}} \cos^2 \frac{(2k+1)\pi}{2n} = n2^{1-n}, \text{ and } \prod_{k=1}^{\frac{n-1}{2}} \sin^2 \frac{2k\pi}{n} = n2^{1-n}.$$

Suppose n is an even positive integer ≥ 4. Then

$$\prod_{k=1}^{\frac{n-2}{2}} \cos^2 \frac{k\pi}{n} = n2^{1-n} \quad \text{and} \quad \prod_{k=1}^{\frac{n-2}{2}} \sin^2 \frac{2k\pi}{n} = n^2 2^{-n}.$$

Suppose n is an even positive integer ≥ 2. Then

$$\prod_{k=1}^{\frac{n-2}{2}} \cos^2 \frac{(2k+1)\pi}{2n} = 2^{1-n}.$$

Proof: For odd $n \geq 3$, we have

$$\prod_{k=1}^{n-1} 2i \cos \frac{k\pi}{n} = F_n(0) = 1,$$

so that

$$2^{n-1} \prod_{k=1}^{\frac{n-1}{2}} \cos^2 \frac{k\pi}{n} = 1.$$

For even $n \geq 4$, let $G_n(x) = \frac{1}{x}F_n(x)$. Then $G_n(0) = n/2$, and

$$\prod_{k=1}^{n-1}\left(x - 2i \cos \frac{k\pi}{n}\right) = x \prod_{\substack{1 \leq k \leq n-1 \\ k \neq n/2}} \left(x - 2i \cos \frac{k\pi}{n}\right) = xG_n(x),$$

so that

$$\prod_{\substack{1 \leq k \leq n-1 \\ k \neq n/2}} 2i \cos \frac{k\pi}{n} = G_n(0) = n/2,$$

and

$$2^{n-2} \prod_{k=1}^{\frac{n-1}{2}} \cos^2 \frac{k\pi}{n} = n/2.$$

Proofs of the other four formulas follow from similar considerations of $L_n(0)$ and $\ell_n(0, 0, 1)$.

Theorem 11: Suppose $a^2 + 4b = 0$. Then, for $n \geq 3$, the roots of the polynomials $U_n(x, z)$, $V_n(x, z)$, and $W_n(x, z)$ are given by the following factorizations.

$$U_n(x, z) = \begin{cases} \displaystyle\prod_{k=1}^{\frac{n-1}{2}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for odd } n \geq 3, \\ \displaystyle\frac{nax}{2} \prod_{k=1}^{\frac{n-2}{2}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for even } n \geq 4. \end{cases}$$

$$V_n(x, z) = \begin{cases} \dfrac{nax}{2}\displaystyle\prod_{k=0}^{\frac{n-3}{2}}\left[cz^2 - bx^2\tan^2\frac{(2k+1)\pi}{2n}\right] & \text{for odd } n \geq 3, \\ \displaystyle\prod_{k=0}^{\frac{n-2}{2}}\left[cz^2 - bx^2\tan^2\frac{(2k+1)\pi}{2n}\right] & \text{for even } n \geq 2. \end{cases}$$

$$W_n(x, z) = \begin{cases} n\displaystyle\prod_{k=1}^{\frac{n-1}{2}}\left[cz^2 - bx^2\cot^2\frac{2k\pi}{n}\right] & \text{for odd } n \geq 3, \\ \dfrac{n^2ax}{4}\displaystyle\prod_{k=1}^{\frac{n-2}{2}}\left[cz^2 - bx^2\cot^2\frac{2k\pi}{n}\right] & \text{for even } n \geq 4. \end{cases}$$

Proof: $U_n(x, z) = F_n(ax, bx^2 + cz^2) = \displaystyle\prod_{k=1}^{n-1}\left(ax - 2i\sqrt{bx^2 + cz^2}\cos\frac{k\pi}{n}\right).$

If n is odd and ≥ 3, then the $n - 1$ roots of $U_n(x, z)$ occur in conjugate pairs, so that

$$\begin{aligned} U_n(x, z) &= \prod_{k=1}^{\frac{n-1}{2}}\left[a^2x^2 + 4(bx^2 + cz^2)\cos^2\frac{k\pi}{n}\right] \\ &= \prod_{k=1}^{\frac{n-1}{2}}\left(-4bx^2\sin^2\frac{k\pi}{n} + 4cz^2\cos^2\frac{k\pi}{n}\right) \\ &= \prod_{k=1}^{\frac{n-1}{2}}4\left(\cos^2\frac{k\pi}{n}\right)\left(cz^2 - bx^2\tan^2\frac{k\pi}{n}\right) \\ &= \prod_{k=1}^{\frac{n-1}{2}}\left(cz^2 - bx^2\tan^2\frac{k\pi}{n}\right) \end{aligned}$$

by Lemma 11.

If n is even and ≥ 4, then the $n - 2$ roots of $U_n(x, z)$ remaining after the root 0 is excluded occur in conjugate pairs, and we find as above that

$$U_n(x, z) = \frac{nax}{2}\prod_{k=1}^{\frac{n-2}{2}}\left(cz^2 - bx^2\tan^2\frac{k\pi}{n}\right).$$

With the help of Lemma 11, the remaining four factorizations are proved in the same way.

Lemma 12: Suppose $a^2 + 4b = 0$. For $n \geq 3$, the transformed generalized Fibonacci cyclotomic polynomial $\mathcal{U}_n(x, z) = \mathcal{F}_n(ax, bx^2 + cz^2)$ is given by

$$\mathcal{U}_n(x,\ z) = \begin{cases} \displaystyle\prod_{\substack{1 \le k \le (n-1)/2 \\ (k,n)=1}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for odd } n \ge 3, \\ \displaystyle\frac{nax}{2} \prod_{\substack{1 \le k \le (n-2)/2 \\ (k,n)=1}} \left(cz^2 - bx^2 \tan^2 \frac{k\pi}{n}\right) & \text{for even } n \ge 4. \end{cases}$$

Proof: This is an obvious consequence of Theorem 11 and the fact that the roots of $\mathcal{F}_n(x,\ z)$ are

$$2i\sqrt{z} \cos \frac{k}{n},\ (k,\ n) = 1,\ 1 \le k \le n - 1.$$

Theorem 12: Suppose a, b, c are integers and $a^2 + 4b = 0$. Except for an integer multiple, for $n \ge 1$, the polynomial $\mathcal{U}_n(x,\ z)$ is irreducible over the ring of integers.

Proof: The proposition is clearly true for $n = 1$ and $n = 2$. Suppose, for $n \ge 3$, that $\mathcal{U}_n(x,\ z) = p(x,\ z)q(x,\ z)$. By Lemma 12 and the irreducibility (since $-b > 0$) of the factors

$$cz^2 - bx^2 \tan^2 \frac{k\pi}{n}$$

over the real number field, $p(x,\ z)$ has the form $P(x,\ z^2)$ and $q(x,\ z)$ has the form $Q(x,\ z^2)$. Thus, putting $r = ax$ and $s = bx^2 + cz^2$, we find

$$\mathcal{F}_n(r,\ s) = P\left(\frac{r}{a},\ \frac{a^2 s - br^2}{a^2 c}\right) Q\left(\frac{r}{a},\ \frac{a^2 s - br^2}{a^2 c}\right).$$

Since $\mathcal{F}_n(r,\ s)$ is irreducible, one of the polynomials P and Q must be constant. But then $p(x,\ z)$ or $q(x,\ z)$ is constant, as desired.

Theorem 13: Suppose $(a,\ b,\ c) = (2, -1,\ 1)$. The Binet formulas for the polynomials U_n, V_n, and W_n are as follows:

$$U_n(x,\ z) = \frac{(x + z)^n - (x - z)^n}{2z}$$

$$V_n(x,\ z) = \frac{(x + z)^n + (x - z)^n}{2}$$

$$W_n(x,\ z) = \begin{cases} \dfrac{1}{x} V_n(x,\ z) & \text{for odd } n, \\ \dfrac{(x + z)^n + (x - z)^n - 2(z^2 - x^2)^{n/2}}{2x} & \text{for even } n. \end{cases}$$

Proof: Let $t_1 = \dfrac{r + \sqrt{r^2 + 4s}}{2}$, $t_2 = \dfrac{r - \sqrt{r^2 + 4s}}{2}$, $t_3 = \sqrt{s}$, $t_4 = -\sqrt{s}$. Putting $r = 2x$ and $s = z^2 - x^2$, the desired formulas follow immediately from the Binet formulas

$$F_n(r,\ s) = \frac{t_1^n - t_2^n}{t_1 - t_2},$$

$$L_n(r,\ s) = t_1^n + t_2^n$$

$$\ell_n(r,\ 0,\ s) = \frac{t_1^n + t_2^n - t_3^n - t_4^n}{t_1 + t_2 - t_3 - t_4}.$$

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GEOMETRIC RECURRENCE RELATION

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1. INTRODUCTION

In a previous paper [1], we considered r, s sequences $\{U_k\}$ and obtained explicit formulations for the general term in powers of r and s. We noted 2 special sequences $\{G_k\}$ and $\{M_k\}$. These are sequences that specialize to the Fibonacci and Lucas sequences where $r = s = 1$.

In this paper, we propose to consider the relationship between r, s recurrence relations and geometric sequences. We give a necessary and sufficient condition on r and s for the recurrence relation to be geometric. We conclude the section by showing how to write any geometric sequence as an r, s recurrence relation.

In the final section, we briefly consider a special Fibonacci sequence. We give an explicit formulation for its general term. We are then able to note when it is a geometric sequence.