$$M_{j-1} = G_{j-1}M_1 + G_{j-2}sM_0 = rG_{j-1} + 2sG_{j-2}.$$

Since $G_j = rG_{j-1} + sG_{j-2}$, it follows that $2sG_{j-2} = 2G_j - 2rG_{j-1}$. We substitute this into the expression for M_{j-1}, and also write the expression for M_j to give the two equations:

$$M_{j-1} = 2G_j - rG_{j-1};$$
$$M_j = rG_j + 2sG_{j-1}.$$

The solutions for G_i and G_{i-1} are

$$G_j = \frac{rM_j + 2sM_{j-1}}{r^2 + 4s} = \frac{M_1M_j + sM_0M_{j-1}}{M_1^2 + sM_0^2}$$

and

$$G_{j-1} = \frac{2M_j - rM_{j-1}}{r^2 + 4s} = \frac{2(rM_{j-1} + sM_{j-2}) - rM_{j-1}}{r^2 + 4s} = \frac{M_1M_{j-1} + sM_0M_{j-2}}{M_1^2 + sM_0^2}.$$

Substituting the results in the expression for U_k of Theorem 4 gives the required expression for this theorem.

The formulation for U_k given in Theorem 5 has been programmed by Robert C. Fitzgerald. He is a senior in Computer Science. We can generate the U_k for specified values of r, s, U_1 and U_0.

Special cases of this result for $e = 0$ and other particular values of r and s will be considered in a future paper.

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THORO'S CONJECTURE AND ALLIED DIVISIBILITY PROPERTY OF LUCAS NUMBERS

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In [3], Thoro made a conjecture that for any prime $p \equiv 3 \pmod 4$, the congruence $F_{2n+1} \equiv 0 \pmod p$ is not solvable where F_{2n+1} is an arbitrary Fibonacci number of odd index. The conjecture has already been proved. In what follows, we give a different proof of this and discuss another problem that arose during this investigation.

Proof: If possible, let the above congruence be true: since $F_{2n+1} = F_n^2 + F_{n+1}^2$ (see [1], p. 56), we get

(1) $$F_n^2 + F_{n+1}^2 \equiv 0 \pmod p$$

Under this hypothesis, it follows that p divides neither F_n nor F_{n+1}. This is justified because if, on the contrary, p divides F_n, then (1) would enable us to conclude that p divides F_{n+1}, forcing us to the invalid result that p divides (F_n, F_{n+1}) or p divides 1. Hence,

$$F_n^2 \equiv -F_{n+1}^2 \pmod{p}.$$

Using Legendre symbol, it means that

$$\left(\frac{-F_{n+1}^2}{p}\right) = 1 \quad \text{or} \quad \left(\frac{-1}{p}\right) = 1.$$

This is not valid, since the prime p is $\equiv 3 \pmod 4$. The required conclusion is now immediate.

Further analysis in regard to divisibility property possessed by Lucas numbers yielded the following theorem.

Theorem: If L_{2n} is an arbitrary Lucas number of even index, then there always exists a prime $p \equiv 3 \pmod 4$ which satisfies the congruence $L_{2n} \equiv 0 \pmod p$.

Proof: Using the result $F_{2n+1} \equiv 1, 2, 5 \pmod 8$ of [3] and the fact that $L_{2n} = F_{2n-1} + F_{2n+1}$ (see [1], p. 56), we obtain $L_{2n} \equiv 2, 3, 4, 6, 7 \pmod 8$. This means that $L_{2n} \not\equiv 1 \pmod 4$. Since the case of L_{2n} being even arises only when $3|n$, we conclude that $L_{6n\pm 2} \equiv 3 \pmod 4$. This means that $L_{6n\pm 2}$ always contains at least one prime factor p with $p \equiv 3 \pmod 4$. In fact, in this case, either this Lucas number is prime of this type or it will contain an odd number of prime factors of this type. For discussion of the case L_{6k}, we first observe that all the members of the family L_{6k} can be obtained from $L_{2^m(6n+3)}$ by choosing suitable values of m and n, where $m = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$. Now, using the fact that

$$L_t | L_s \text{ iff } s = (2k - 1)t$$

(see [1], p. 40), we get

$$L_{2^m} | L_{2^m(6n+3)}.$$

Since $(2^m, 3) = 1$, by previous discussion, there always exists a prime $p \equiv 3 \pmod 4$ such that $p | L_{2^m}$, which implies that $p | L_{2^m(6n+3)}$ and the proof is complete. It is easy to verify that $3|L_6$, $7|L_{12}$, $3|L_{18}$, $47|L_{24}$ and so on. For a strong result, namely $2 \cdot 3^k | L_{2 \cdot 3^k}$, refer to [2].

Corollary: L_{6n} contains an even number of prime factors p where $p \equiv 3 \pmod 4$.

Proof: From the well-known identities (see [1], p. 56), we have

$$L_{2n} = F_{n-1}^2 + 2F_n^2 + F_{n+1}^2,$$

which yields

$$L_{6n} = F_{3n-1}^2 + 2F_{3n}^2 + F_{3n+1}^2.$$

Since F_{3n} is even whereas F_{3n-1} and F_{3n+1} are odd, we have $F_{3n-1}^2 \equiv 1 \pmod 8$, $F_{3n+1}^2 \equiv 1 \pmod 8$, and $2F_{3n}^2 \equiv 0 \pmod 8$. Therefore, $L_{6n} \equiv 2 \pmod 8$ or $L_{6n} = 2(4\alpha + 1)$ for a suitable α.

From the above theorem, we have the existence of at least one prime $p \equiv 3 \pmod 4$ such that $p | L_{6n}$. We conclude that L_{6n} must have an even number of such factors for justifying the odd factor $(4\alpha + 1)$ stated above.

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A CLASS OF SOLUTIONS OF THE EQUATION $\sigma(n) = 2n + t$

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INTRODUCTION

Let the nondeficient natural number n satisfy

$$f(n) = t, \tag{1}$$

where $f(n) = \sigma(n) - 2n$, and t is a given nonnegative integer. Clearly, (1) is equivalent to

$$\sigma(n) = 2n + t. \tag{1*}$$

Definition 1: m is acceptable with respect to n if m is a nondeficient proper divisor of n.

Definition 2: n is primitive if no number is acceptable with respect to n; otherwise, n is nonprimitive.

Remark 1: Primitive nondeficient numbers were defined by L. E. Dickson [3], p. 413.

If $t = 0$ in (1), then n is called perfect. It is known that when n is perfect:

(a) if n is even, then $n = 2^{p-1}(2^p - 1)$ where $2^p - 1$ is prime (Euclid-Euler);
(b) if n is odd, then n has at least 8 distinct prime factors [4] and exceeds 10^{50} [5];
(c) n is primitive.

If $t = 1$ in (1), then n is called quasiperfect [2]. It is known that if n is quasiperfect, then:

(a) n is odd and primitive [2];
(b) n has at least 6 distinct prime factors and exceeds 10^{30} [6].

On the other hand, for $t = 3$, by inspection we obtain the nonprimitive solution $n = 18$. This suggests that nonprimitive solutions of (1), when they exist, are more easily obtained than primitive ones.

In this article, we shall determine the set of all nonprimitive solutions of (1) for each t such that $2 \le t \le 100$. Theorem 1 states that Table 5 contains all such solutions for the given range of values of t.

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