# A THEOREM CONCERNING HEPTAGONAL NUMBERS

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In this paper, we show that there are an infinite number of heptagonal numbers which are, at the same time, the sums and differences of distinct heptagonal numbers. Similar results have been found for triangular numbers [1] and pentagonal numbers [2].

The heptagonal numbers are given by $h_n = n(5n - 3)/2$, $n = 1, 2, 3, \ldots$, where $h_n - h_{n-1} = 5n - 4$. Heptagonal numbers are represented geometrically by regular heptagons homothetic with respect to one of the vertices and containing $2, 3, 4, \ldots, n$ points at equal distances along each side. The sum of all the points for a given $n$ yields $h_n$. Both Dickson [3] and LeVeque [4] provide reviews concerning heptagonal and related figurate numbers.

Our analysis starts with the observation from a table of $h_n$ values [5] that

$$h_{17} = h_6 + h_{16},\ h_{58} = h_{11} + h_{57},\ \text{and}\ h_{124} = h_{16} + h_{123}.$$

Note that each of these equations is of the form $h_m = h_{5k+1} + h_{m-1}$.

Since $h_{5k-1} = (125k^2 + 35k + 2)/2$, setting

$$h_m - h_{m-1} = 5m - 4 = h_{5k+1} = (125k^2 + 35k + 2)/2,$$

we have

$$m = (125k^2 + 35k + 10)/10.$$

An induction proof shows that $m$ is an integer for all integers $k$. This leads us to:

<u>*Theorem 1*</u>: For any integer $k \geqq 1$,

$$h_{\frac{125k^2+35k\ \ 10}{10}} = h_{5k+1} + h_{\frac{125k^2+35k}{10}}.$$

Now consider the subset of heptagonal numbers in Theorem 1 which yields

$$(*)\qquad h_{\frac{125(5k)^2+35(5k)+10}{10}} = h_{5(5k)+1} + h_{\frac{125(5k)^2+35(5k)}{10}}.$$

The LHS of (*) is equal to

$$(**)\qquad (9765625k^4 + 1093750k^3 + 74375k^2 + 2450k + 40)/40.$$

But suppose that $h_s - h_{s-1} = 5s - 4 = (**)$, so that we have

$$s = (9765625k^4 + 1093750k^3 + 74375k^2 + 2450k + 200)/200.$$

An induction proof shows that $s$ is an integer for all positive integers $k$. Therefore, we have our major result,

<u>*Theorem 2*</u>: For any integer $k \geqq 1$,

$$h_{\frac{3125k^2+175k+10}{10}} = h_{25k+1} + h_{\frac{3125k^2+175k}{10}}$$

and

$$h_{\frac{3125k^2+175k+10}{10}} = h_{\frac{9765625k^4+1093750k^3+74375k^2+2450k+200}{200}} - h_{\frac{9765625k^4+1093750k^3+74375k^2+2450k}{200}}.$$

Since these results hold for all integers $k \geq 1$, we see that there are an infinite number of heptagonal numbers which are, at the same time, the sums and differences of distinct heptagonal numbers. Q.E.D.

For $k$ = 1, 2, and 3, respectively, Theorem 2 yields

$$h_{331} = h_{26} + h_{330} = h_{54682} - h_{54681},$$
$$h_{1286} = h_{51} + h_{1285} = h_{826513} - h_{826512},$$
$$h_{2866} = h_{76} + h_{2865} = h_{4106119} - h_{4106118}.$$

Verification is straightforward, if tedious. The list may be continued as desired.

Triangular, pentagonal, and heptagonal numbers all have the property exemplified by Theorem 2 for heptagonal numbers. Therefore, the question naturally arises as to whether either nonagonal or any or all other "odd number of sides" figurate numbers have the property. This conjecture is under investigation.

## REFERENCES


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# AN ALTERNATE REPRESENTATION FOR CÉSARO'S FIBONACCI-LUCAS IDENTITY


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E. Césaro's symbolic Fibonacci-Lucas identity $(2u+1)^n = u^{3n}$ allows us, after the binomial expansion has been performed, to use the powers as either Fibonacci or Lucas subscripts and obtain useful identities [1]. These have appeared many times in the literature, and most recently have been the subject of a problem [2].

Use of the identity enables us to provide a finite sum for $F_{3n}$ (or $L_{3n}$) which is a linear combination of terms from $F_0$ (or $L_0$) to $F_n$ (or $L_n$) inclusive. For example, we may derive

$$4L_2 + 4L_1 + L_0 = L_6$$

or, with algebraic effort, we obtain

$$16F_4 + 32F_3 + 24F_2 + 8F_1 + F_0 = F_{12}.$$

In this note, I show that

$$\sum_{r=0}^{n} 2^{n-r}\binom{n}{n-r}F_{n-r} = F_{3n} \tag{1}$$