Since these results hold for all integers $k \geqq 1$, we see that there are an infinite number of heptagonal numbers which are, at the same time, the sums and differences of distinct heptagonal numbers. Q.E.D.

For $k = 1$, 2, and 3, respectively, Theorem 2 yields

$$h_{331} = h_{26} + h_{330} = h_{54682} - h_{54681},$$
$$h_{1286} = h_{51} + h_{1285} = h_{826513} - h_{826512},$$
$$h_{2866} = h_{76} + h_{2865} = h_{4106119} - h_{4106118}.$$

Verification is straightforward, if tedious. The list may be continued as desired.

Triangular, pentagonal, and heptagonal numbers all have the property exemplified by Theorem 2 for heptagonal numbers. Therefore, the question naturally arises as to whether either nonagonal or any or all other "odd number of sides" figurate numbers have the property. This conjecture is under investigation.

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AN ALTERNATE REPRESENTATION FOR CÉSARO'S FIBONACCI-LUCAS IDENTITY

HARVEY J. HINDIN

Polymathic Associates, 5 Kinsella St., Dix Hills, NY 11746

E. Césaro's symbolic Fibonacci-Lucas identity $(2u+1)^n = u^{3n}$ allows us, after the binomial expansion has been performed, to use the powers as either Fibonacci or Lucas subscripts and obtain useful identities [1]. These have appeared many times in the literature, and most recently have been the subject of a problem [2].

Use of the identity enables us to provide a finite sum for F_{3n} (or L_{3n}) which is a linear combination of terms from F_0 (or L_0) to F_n (or L_n) inclusive. For example, we may derive

$$4L_2 + 4L_1 + L_0 = L_6$$

or, with algebraic effort, we obtain

$$16F_4 + 32F_3 + 24F_2 + 8F_1 + F_0 = F_{12}.$$

In this note, I show that

$$\sum_{r=0}^{n} 2^{n-r}\binom{n}{n-r}F_{n-r} = F_{3n} \tag{1}$$

is entirely equivalent to the Césaro identity where F_{3n} may be replaced by L_{3n}. This is of inherent interest and allows the direct determination of the multiplying coefficients for the finite sum without requiring a binomial expansion. For example, we may write, by inspection of (1), that

$$\begin{aligned}F_{24} = {} & 2^8\binom{8}{8}F_8 + 2^7\binom{8}{7}F_7 + 2^6\binom{8}{6}F_6 + 2^5\binom{8}{5}F_5 + 2^4\binom{8}{4}F_4 \\ & + 2^3\binom{8}{3}F_3 + 2^2\binom{8}{2}F_2 + 2^1\binom{8}{1}F_1 + 2^0\binom{8}{0}F_0.\end{aligned} \tag{2}$$

To derive or "discover" (1), construct, starting with $n = 0$, a Pascal triangle form of the coefficient multipliers of the LHS of the Césaro identity. This is shown in Figure 1a.

1						$n = 0$
2	2					$n = 1$
4	4	1				$n = 2$
8	12	6	1			$n = 3$
16	32	24	8	1		$n = 4$
32	80	80	40	10	1	$n = 5$
.	.	.	.	.	.	.

Fig. 1a. Coefficient Multiplier Array for LHS of Césaro Identity

Note that this array may be written in the form shown in Figure 1b.

1 (1)						$n = 0$
2 (1)	1 (1)					$n = 1$
4 (1)	2 (2)	1 (1)				$n = 2$
8 (1)	4 (3)	2 (3)	1 (1)			$n = 3$
16 (1)	8 (4)	4 (6)	2 (4)	1 (1)		$n = 4$
32 (1)	16 (5)	8 (10)	4 (10)	2 (5)	1 (1)	$n = 5$
.	.	.	.	.	.	.

Fig. 1b. Alternate Coefficient Multiplier Array for LHS of Césaro Identity

It may be seen that Figure 1b is the usual Pascal array with power of 2 multipliers. Indeed the rth term in the nth row, where $0 \leqq r \leqq n$ is given by $2^{n-r}\binom{n}{n-r}$. It follows directly that we may multiply each coefficient in row n by its corresponding Fibonacci or Lucas term, sum them, and set the result equal to the appropriate RHS of the Césaro identity to obtain

$$\sum_{r=0}^{n} 2^{n-r}\binom{n}{n-r}F_{n-r} = F_{3n}$$

for the Fibonacci case. Q.E.D.

This result, which clearly holds for Lucas numbers also, has not been noted previously as the equivalent of the Cesaro identity. The discovery method of derivation used here is particularly satisfying.

We may note in passing that the row sum in Figure 1b is given by

$$\sum_{r=0}^{n} 2^{n-r}\binom{n}{n-r} = 3^n. \tag{3}$$

Also, the right-rising diagonal generates the series 1, 2, 5, 12, 29, 70, ... given by $R_n = 2R_{n-1} + R_{n-2}$, and the left-rising diagonal yields 1, 1, 3, 5, 11, 21, ... given by $L_n = L_{n-1} + 2L_{n-2}$. Other properties of the array may be found by the reader.

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ON THE MATRIX APPROACH TO FIBONACCI NUMBERS AND THE FIBONACCI PSEUDOPRIMES

JACK M. POLLIN

United States Military Academy, West Point, NY

AND

I. J. SCHOENBERG

Mathematics Research Center, University of Wisconsin-Madison, WI 53706

INTRODUCTION

We consider sequences (x_n) of integers satisfying for all n the recurrence relation

$$x_{n+1} = x_n + x_{n-1}. \tag{1}$$

The x_n are uniquely defined if we prescribe the elements of the "initial vector" (x_0, x_1). On choosing $(x_0, x_1) = (0, 1)$, we obtain the *Fibonacci numbers* $x_n = F_n$, while the choise $(x_0, x_1) = (2, 1)$ gives the *Lucas numbers* $x_n = L_n$.

In [3], V. E. Hoggatt, Jr., and Marjorie Bicknell discuss the following conjecture of K. W. Leonard (unpublished).

Conjecture 1: *We have the congruence*

$$L_n \equiv 1 \pmod{n}, \quad (n > 1) \tag{2}$$

if and only if n is a prime number.

Among the many interesting results of [3], we single out the following:

Theorem 1: *The "if" part of Conjecture 1 is correct; i.e.,*

$$L_p \equiv 1 \pmod{p}, \text{ where } p \text{ is a prime.} \tag{3}$$

Theorem 2: *The "only if" part of Conjecture 1 is wrong, as shown by the congruence*

$$L_{705} \equiv 1 \pmod{705}, \tag{4}$$

while $705 = 3 \cdot 5 \cdot 47$ *is composite.*

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