This is in set A_2, because T_{3n+3} is in A_1. Thus, since $(T_{3n+4} + T_{3n+3})$ and $(T_{3n+5} - T_{3n+3})$ are both in A_2, $T_{3n+5} + T_{3n+4}$ has a representation as the sum of two elements from set A_2.

Next, consider

$$\begin{aligned} &T_{3n+4} + T_{3n+3} - (T_{3n+1} + T_{3n}) \\ &= T_{3n+4} + T_{3n+3} + T_{3n+2} - (T_{3n+2} + T_{3n+1} + T_{3n}) \\ &= T_{3n+5} - T_{3n+3}, \end{aligned}$$

which we have seen to lie in A_2, so that

$$(T_{3n+5} - T_{3n+3}) + (T_{3n+1} + T_{3n}) = T_{3n+4} + T_{3n+3}$$

is the sum of two integers from A_2, since both are in A_2. This completes the proof.

If $n \neq T_m$ or $n \neq T_s + T_{s+1}$, then n has a representation as the sum of two elements from the same set. If $n = T_s + T_{s+1}$, then if $n = T_{3m+1} + T_{3m+2}$, both T_{3m+1} and T_{3m+2} appear in A_1, and n has a representation as the sum of two elements from A_1. If $n = T_{3m+2} + T_{3m+3}$ or $n = T_{3m} + T_{3m+1}$, then each has a sum of two elements from A_2.

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THE NUMBER OF MORE OR LESS "REGULAR" PERMUTATIONS

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Let us call S_{m+1} the set of all permutations of the integers $\{1, 2, \ldots, m + 1\}$. Any permutation α from S_{m+1} may be decomposed into b blocks $B_1, B_2, \ldots, B_b$ defined by the following property: each block consists of integers increasing unit by unit, and no longer block has the same property.

<u>*Example*</u>: $m = 8$, $\alpha = 314562897$; there are $b = 6$ blocks:

$$B_1 = 3,\ B_2 = 1,\ B_3 = 456,\ B_4 = 2,\ B_5 = 89,\ B_6 = 7.$$

The lengths of the blocks form a b-composition q of $m + 1$ (see [1]); in the above example, $q = (1, 1, 3, 1, 2, 1)$.

If $\alpha(i)$ is the ith integer in α, $\alpha(i)$ and $\alpha(i + 1)$ belong to the same block iff $\alpha(i + 1) = \alpha(i) + 1$; let us call the number of i's satisfying this condition the regularity r of α. Obviously $b + r = m + 1$, so that b and r are equivalent descriptive parameters of α. The greatest possible regularity is $r = m$; it occurs iff α is the identical permutation. The smallest possible regularity is $r = 0$; it occurs iff $q = (1, 1, 1, \ldots, 1)$; we shall call the corresponding permutations "irregular permutations," and look for their number. More generally, we shall call $U(m, r)$ the subset of S_{m+1} consisting of the permutations of regularity r, and $u(m, r)$ the cardinality of $U(m, r)$. We know already that $u(m, m) = 1$ and that

$$\sum_{r=0}^{m} u(m, r) = (m + 1)! \tag{1}$$

Setting $u(m, 0) = u$, we shall first show that

$$u(m, r) = \binom{m}{r} u_{m-r}. \tag{2}$$

Let us start from a permutation α of regularity r, i.e., of $b = m - r + 1$ blocks. Besides their order of appearance in α, there is an "order of increasing values" of the blocks; in that order, the smallest block in the above example is 1 ($=B_2$), then comes 2 ($=B_4$), then 3 ($=B_1$), then 456 ($=B_3$), then 7 ($=B_6$), and finally, 89 ($=B_5$). If we relabel the blocks according to their place in the latter order, and if we list them by order of appearance in α, we obtain a permutation p of $\{1, 2, \ldots, b\}$; in the above example, $p = (314265)$.

Necessarily, this permutation p is an irregular one, since, if it had two consecutive integers at two consecutive places, it would mean that the corresponding blocks in α could be merged into a single block, which is contradictory with the definition of the "blocks."

Let us start now from the pair (p, q), where p is any irregular permutation of $\{1, 2, \ldots, m - r + 1\}$ and q is any $(m - r + 1)$-composition of $m + 1$:

$$p = (p_1, p_2, \ldots, p_b),$$
$$q = (q_1, q_2, \ldots, q_b).$$

If $p_i = p(i) = 1$, transform p by replacing p_i by a block $(123 \ldots q_i)$; if $p(j) = 2$, replace p_j by a block $(q_i + 1, q_i + 2, \ldots, q_i + q_j)$, and so on, until p is finally transformed into a permutation α of $\{1, 2, \ldots, m + 1\}$.

This procedure defines in fact a $(1 - 1)$-correspondence between the set $U(m, r)$ and the set of pairs (p, q) consisting of an irregular permutation p of $\{1, 2, \ldots, m - r + 1\}$ and a $(m - r + 1)$-composition q of $m + 1$. Since it is well known that the number of u-compositions of v is $\binom{v-1}{u-1}$, we can conclude that

$$u(m, r) = u_{m-r}\binom{m}{m-r},$$

which proves (2).

Inverting (1) after replacement of $u(m, r)$ by its expression (2), we obtain

$$u = \sum_{r=0}^{m} (-1)^r \binom{m}{r} (m + 1 - r)!,$$

which may be written

$$u_m = \Delta^m 1!. \tag{3}$$

This enables us to calculate u_m for moderate values of m:

$m =$	0	1	2	3	4	5	6	...
$u_m =$	1	1	3	11	53	309	2119	...

For larger values of m, it is convenient to use recursion formulas with positive terms only, which will be connected with a closer investigation of irregular permutations.

If we start from one of the u_m permutations belonging to $U(m, 0)$, say α, and if we delete $m+1$ in α, the remaining permutation β of $\{1, 2, \ldots, m\}$ may be irregular or not, and, in fact, will be of regularity either 0 or 1. Conversely, the whole set $U(m, 0)$ can be reconstructed by the reinsertion of integer $m+1$ either at some suitable place of an irregular permutation β or at the only suitable place of a permutation β of regularity 1.

If β is irregular, there are $m + 1$ conceivable places for insertion of integer $m + 1$, but one and only one of them, namely the place immediately after integer m, is not suitable. The number of corresponding possibilities is thus mu_{m-1}.

If β is of regularity 1, the number of possibilities for β is given by formula (2), substituting $m - 1$ for m and 1 for r, which yields $(m-1)u_{m-2}$; integer $m + 1$ must then be inserted between the only two consecutive integers of β.

Finally,

$$u_m = mu_{m-1} + (m - 1)u_{m-2}, \tag{4}$$

which provides an easier calculation of the sequence.

A numerical table of $u(m, r)$ is readily formed from the knowledge of u_m and formula (2):

	$r = 0$	1	2	3	4
$m = 0$	1				
1	1	1			
2	3	2	1		
3	11	9	3	1	
4	55	44	18	4	1

The following properties are easy to verify:

(1°) Column $r = 1$ consists of the "rencontres" numbers (see [2]). The numbers of columns 0 and 1 appear in [3], but without reference to their enumerative meaning.

(2°) The Blissard generating function [2] of column 0,

$$y(x) = \sum_{m=0}^{+\infty} u_m \frac{x^m}{m!},$$

satisfies the differential equation

$$y'(1 - x) = y(1 + x),$$

since (4) may be written

$$u_{m+1} - mu_m = u_m + mu_{m-1}.$$

Elementary integration yields

$$y = y_0 = e^{-x}(1 - x)^{-2}.$$

(3°) The Blissard generating function y_r of column r is given by use of (2):

$$\sum_m \binom{m}{r} u_{m-r}\frac{x^m}{m!} = \frac{x^r}{r!}\sum_m u_{m-r}\frac{x^{m-r}}{(m-r)!},$$

so that

$$y_r = e^{-x}(1-x)^{-2}x^r/r!.$$

(4°) The sum $\sum_{r=0}^{+\infty} y_r$ is $(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots$, which confirms that the coefficient of $x^m/m!$ is $(m+1)!$.

(5°) According to (3), the ratio $u_m/(m+1)!$ is equal to

$$1 - \binom{m}{1}\frac{1}{m+1} + \binom{m}{2}\frac{1}{(m+1)m} - \cdots + (-1)^r\binom{m}{r}\frac{1}{(m+1)_r} + \cdots$$

As m increases, with fixed r, the general term of this sum tends toward $(-1)^r/r!$; it follows that the sum itself tends toward e^{-1}, which is the limiting proportion of irregular permutations.

(6°) Using (2), it appears that

$$\frac{u(m, r)}{(m+1)!} = \frac{u_{m-r}}{(m-r+1)!}\cdot\frac{m-r+1}{r!(m+1)}.$$

As m increases, the second member tends toward $e^{-1}/r!$. The latter result means that, if a permutation is chosen at random in S_{m+1} and if m increases, the limiting probability distribution of its regularity is a Poisson distribution with mean 1.

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STAR POLYGONS, PASCAL'S TRIANGLE, AND FIBONACCI NUMBERS

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In recent years, there has been some flurry of excitement over the relationship between the complexity of a graph, i.e., the number of distinct spanning trees in a graph, and the Fibonacci and Lucas numbers [1, 2]. In this note, I shall demonstrate a relationship, although incomplete, between the Fibonacci numbers and the star polygons. My hope is to spur further research into the connection between nonplanar graphs and their enumeration from number theory.