ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN STATE COLLEGE, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate, signed sheets within two months of publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-320 *Proposed by Paul S. Bruckman, Concord, CA.*

Let

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},\ \mathrm{Re}(s) > 1, \text{ the Riemann Zeta function.}$$

Also, let

$$H_n = \sum_{k=1}^{n} k^{-1},\ n = 1, 2, 3, \ldots, \text{ the harmonic sequence.}$$

Show that

$$\sum_{n=1}^{\infty} \frac{H_n}{n^2} = 2\zeta(3).$$

H-321 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA.*

Establish the identity

$$\begin{aligned} F^6_{n+14r} &+ F^6_n - (L_{12r} + L_{8r} + L_{4r} - 1)(F^6_{n+12r} + F^6_{n+2r}) \\ &+ (L_{20r} + L_{16r} + L_{4r} + 3)(F^6_{n+10r} + F^6_{n+4r}) \\ &- (L_{24r} - L_{20r} + L_{12r} + 2L_{8r} - 1)(F^6_{n+8r} + F^6_{n+6r}) \\ &= 40(-1)^n \prod_{i=1}^{3} F^2_{2ri}. \end{aligned}$$

SOLUTIONS

Δ Dawn

H-294 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA. (Vol. 16, No. 6, December 1978)*

Evaluate

$$\Delta = \begin{vmatrix} F_{2r+1} & F_{6r+3} & F_{10r+5} & F_{14r+7} & F_{18r+9} \\ F_{4r+2} & -F_{12r+6} & F_{20r+10} & -F_{28r+14} & F_{36r+18} \\ F_{6r+3} & F_{18r+9} & F_{30r+15} & F_{42r+21} & F_{54r+27} \\ F_{8r+4} & -F_{24r+12} & F_{40r+20} & -F_{56r+28} & F_{72r+36} \\ F_{10r+5} & F_{30r+15} & F_{50r+25} & F_{70r+35} & F_{90r+45} \end{vmatrix}$$

Solution by the proposer.

After simplification,

$$\Delta = F_{2r+1}F_{6r+3}F_{10r+5}F_{14r+7}F_{18r+9}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ L_{2r+1} & -L_{6r+3} & L_{10r+5} & -L_{14r+7} & L_{18r+9} \\ L_{4r+2} & L_{12r+6} & L_{20r+10} & L_{28r+14} & L_{36r+18} \\ L_{6r+3} & -L_{18r+9} & L_{30r+15} & -L_{42r+21} & L_{54r+27} \\ L_{8r+4} & L_{24r+12} & L_{40r+20} & L_{56r+28} & L_{72r+36} \end{vmatrix}$$

$$= F_{2r+1}F_{6r+3}F_{10r+5}F_{14r+7}F_{18r+9}(L_{6r+3} + L_{2r+1})(L_{10r+5} - L_{2r+1})$$
$$\cdot\ (L_{14r+7} + L_{2r+1})(L_{18r+9} - L_{2r+1})(L_{10r+5} + L_{6r+3})(L_{14r+7} - L_{6r+3})$$
$$\cdot\ (L_{18r+9} + L_{6r+3})(L_{14r+7} + L_{10r+5})(L_{18r+9} - L_{10r+5})(L_{18r+9} + L_{14r+7})$$
$$= 5^{10}F^5_{2r+1}F^4_{4r+2}F^4_{6r+3}F^3_{8r+4}F^3_{10r+5}F^2_{12r+6}F^2_{14r+7}F_{16r+8}F_{18r+9}$$
$$= 5^{10}F^5_wF^4_{2w}F^4_{3w}F^3_{4w}F^3_{5w}F^2_{6w}F^2_{7w}F_{8w}F_{9w}, \text{ where } w = 2r + 1.$$

More Identities

H-295 *Proposed by G. Wulczyn, Bucknell University, Lewisburg, PA.* *(Vol. 17, No. 1, February 1979)*

Establish the identities

(a) $F_kF^2_{k+6r+3} - F_{k+8r+4}F^2_{k+2r+1} = (-1)^{k+1}F^3_{2r+1}L_{2r+1}L_{k+4r+2}$

and

(b) $F_kF^2_{k+6r} - F_{k+8r}F^2_{k+2r} = (-1)^{k+1}F^3_{2r}L_{2r}L_{k+4r}.$

Solution by the proposer.

(a) $F_kF^2_{k+6r+3} - F_{k+8r+4}F^2_{k+2r+1}$

$$= \frac{1}{5\sqrt{5}}\Big\{(\alpha^k - \beta^k)[\alpha^{2k+12r+6} + \beta^{2k+12r+6} + 2(-1)^k]$$
$$- (\alpha^{k+8r+4} - \beta^{k+8r+4})[\alpha^{2k+4r+2} + \beta^{2k+4r+2} + 2(-1)^k]\Big\}$$
$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\Big\{\alpha^{k-4r-2}(\alpha^{16r+8} - 2\alpha^{4r+2} + 2\alpha^{12r+6} - 1)$$
$$- \beta^{k-4r-2}(\beta^{16r+8} + 2\beta^{12r+6} - 2\beta^{4r+2} - 1)\Big\}$$
$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\Big\{\alpha^{k-4r-2}(\alpha^{4r+2} - 1)(\alpha^{4r+2} + 1)^3 - \beta^{k-4r-2}(\beta^{4r+2} - 1)(\beta^{4r+2} + 1)^3\Big\}$$
$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\Big\{\alpha^{k-4r-2}\alpha^{2r+1}(\alpha^{2r+1} + \beta^{2r+1})(\alpha^{2r+1})^3(\alpha^{2r+1} - \beta^{2r+1})^3$$
$$+ \beta^{k+4r+2}(\alpha^{2r+1} + \beta^{2r+1})(\alpha^{2r+1} - \beta^{2r+1})^3\Big\}$$
$$= (-1)^{k+1}F^3_{2r+1}L_{2r+1}L_{k+4r+2}.$$

(b) $F_kF^2_{k+6r} - F_{k+8r}F^2_{k+2r}$

$$= \frac{1}{5\sqrt{5}}\Big\{(\alpha^k - \beta^k)[\alpha^{2k+12r} + \beta^{2k+12r} + 2(-1)^{k+1}]$$
$$- (\alpha^{k+8r} - \beta^{k+8r})[\alpha^{2k+4r} + \beta^{2k+2r} + 2(-1)^{k+1}]\Big\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\left\{\alpha^{k+4r}(\alpha^{16r} - 2\alpha^{12r} + 2\alpha^{4r} - 1) - \beta^{k-4r}(\beta^{16r} - 2\beta^{12r} + 2\beta^{4r} - 1)\right\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\left\{\alpha^{k-4r}(\alpha^{4r} - 1)^3(\alpha^{4r} + 1) - \beta^{k-4r}(\beta^{4r} - 1)^3(\beta^{4r} + 1)\right\}$$

$$= \frac{(-1)^{k+1}}{5\sqrt{5}}\left\{\alpha^{k+4r}(\alpha^{2r} - \beta^{2r})^3(\alpha^{2r} + \beta^{2r}) + \beta^{k+4r}(\alpha^{2r} - \beta^{2r})^3(\alpha^{2r} + \beta^{2r})\right\}$$

$$= (-1)^{k+1}F_{2r}^3 L_{2r} L_{k+4r}.$$

Also solved by P. Bruckman.

Bracket Your Answer

H-296 *Proposed by C. Kimberling, University of Evansville, Evansville, IN. (Vol. 17, No. 1, February 1979)*

Suppose x and y are positive real numbers. Find the least positive integer n for which

$$\left[\frac{x}{n+y}\right] = \left[\frac{x}{n}\right]$$

where $[z]$ denotes the greatest integer less than or equal to z.

Partial solution by the proposer.

Solution for the Special Case $y = 1$:

Let $m = [\sqrt{x}]$ and $A = \begin{cases} \frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} & \text{if } m^2 \le x \le m^2 + m - 1 \\ 1 + \sqrt{(m+1)^2 - x} & \text{if } m^2 + m \le x \le m^2 + 2m \end{cases}$

Then the least positive integer n satisfying $\left[\frac{x}{n+1}\right] = \left[\frac{x}{n}\right]$ is given by

$$n = \begin{cases} m - 1 + A & \text{if } A \text{ is an integer} \\ m + [A] & \text{otherwise.} \end{cases}$$

Proof: First suppose $m^2 \le x \le m^2 + m - 1$, where $m = [\sqrt{x}]$. Let $L = x - m^2$. Then writing $k = n - m$ gives

$$\frac{x}{n} = \frac{x}{m+k} = \frac{m^2 + L}{m+k} = m - k + \frac{k^2 + L}{m+k}.$$

Similarly

$$\frac{x}{n+1} = m - k - 1 + \frac{(k+1)^2 + L}{m+k+1}.$$

The least k satisfying $\frac{(k+1)^2 + L}{m+k+1} \ge 1$ is easily found to satisfy

$$k \ge -\frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x}.$$

Thus, for

$$k = \begin{cases} -\frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} & \text{if this is an integer} \\ \frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} & \text{otherwise} \end{cases}$$

we find that $k < \frac{1}{2} + \sqrt{\left(m + \frac{1}{2}\right)^2 - x} = \frac{1}{2} + \sqrt{m - L + \frac{1}{4}}$, so that

$$\left(k - \frac{1}{2}\right)^2 < n - L + \frac{1}{4} \quad \text{and} \quad \frac{k^2 + L}{m + k} > 1.$$

Consequently, $\left[\frac{x}{n}\right] = m - k$. Furthermore, if

$$\frac{(k + 1)^2 + L}{m + k + 1} \geq 2$$

then $\frac{x}{1 + n} \geq m - k + 1$, contrary to $\frac{x}{n + 1} < \frac{x}{n} < m - k$. This shows that

$$\frac{(k + 1)^2 + L}{m + k + 1} < 2$$

so that $\left[\frac{x}{1 + n}\right] = m - k$.

If $n' < n$, then for $n = m + k'$ we have $k' < k$; by definition of k this implies

$$\frac{(k' + 1)^2 + L}{m + k' + 1} < 1$$

so that

$$\frac{x}{1 + n'} = m - k' - 1 + \frac{(k' + 1)^2 + L}{m + k' + 1} \quad \text{and} \quad \left[\frac{x}{1 + n'}\right] \leq m - k' - 1.$$

On the other hand, $\frac{x}{n'} = m - k' + \frac{k'^2 + L}{m + k'}$, so that $\left[\frac{x}{n'}\right] \geq m - k'$. This shows that n is indeed the *least* positive integer for which $\left[\frac{x}{n + 1}\right] = \left[\frac{x}{n}\right]$.

Now suppose $m^2 + m \leq x \leq m^2 + 2m$, where again $m = [\sqrt{x}]$. Let

$$L = x - m^2 - m \quad \text{and} \quad k = n - m.$$

An argument analogous to that above shows that the least k for which

$$\left[\frac{x}{m + k}\right] = \left[\frac{x}{m + k + 1}\right]$$

is given by

$$k = \begin{cases} \sqrt{(m + 1)^2 - x} & \text{if this is an integer} \\ 1 + \left[\sqrt{(m + 1)^2 - x}\right] & \text{otherwise.} \end{cases}$$

The solution stated above now follows from $k = n - m$.

Note: It appears likely that for any y, at least for any positive *integer* y, the solution can be written in the form $[x/j] + 1$, where j is an integer.

Also solved by C. B. A. Peck.

The Limit

H-297 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA. (Vol. 17, No. 1, February 1979)*

Let $P_0 = P_1 = 1$, $P_n(\lambda) = P_{n-1}(\lambda) - \lambda P_{n-2}(\lambda)$. Show

$$\lim_{n \to \infty} P_{n-1}(\lambda)/P_n(\lambda) = (1 - \sqrt{1 - 4\lambda})/2\lambda = \sum_{n=0}^{\infty} C_{n+1} x^n,$$

where C_n is the nth Catalan number. Note that the coefficients of $P_n(\lambda)$ lie along the rising diagonals of Pascal's triangle with alternating signs.

Solution by Paul S. Bruckman, Concord, CA.

The characteristic polynomial of the P_n's is $x^2 - x + \lambda = (x - u)(x - v)$, where

$$u = u(\lambda) = \frac{1}{2}(1 + \sqrt{1 - 4\lambda}), \quad v = v(\lambda) = \frac{1}{2}(1 - \sqrt{1 - 4\lambda}).$$

It follows readily from the initial conditions that

$$P_n(\lambda) = (u^{n+1} - v^{n+1})/(u - v), \quad n = 0, 1, 2, \ldots .$$

Although it is not stated in the problem, we assume that $|\lambda| < 1/4$, to avoid possible problems of convergence. Being acquainted to some degree with the proposer of the problem, it is nearly safe to say that he did not intend the problem to involve a rigorous treatment with λ ranging over all admissible values, but rather a formal result valid for "nice" values of λ. Moreover, we assume λ is real.

Let $r_n = P_{n-1}(\lambda)/P_n(\lambda)$, and $f(\lambda) = u(\lambda)/v(\lambda)$. Since $uv = \lambda$,

$$f(\lambda) = u^2/\lambda = (u - \lambda)/\lambda = u(\lambda)/\lambda - 1.$$

Also

$$r_n(\lambda) = (u^n - v^n)/(u^{n+1} - v^{n+1}) = \frac{(u/v)^n - 1}{v\{(u/v)^{n+1} - 1\}} = \frac{f^n - 1}{v(f^{n+1} - 1)}.$$

If we consider the graph of f, we see that the graph has asymptotes at $\lambda = 0$ and at $f = -1$; however, the latter asymptote is approached only as $\lambda \to -\infty$, and we exclude this possibility, by hypothesis. If $\lambda > 0$, clearly $u > v$. If $\lambda < 0$, then $u > 1$, $v < 0$, and $f < -1$. It follows that, if $|\lambda| < 1/4$, then $|f| > 1$.

Hence, $r = \lim\limits_{n\to\infty} r_n$ exists and

$$r = 1/vF = 1/u = v/\lambda = \frac{1 - \sqrt{1 - 4\lambda}}{2\lambda}.$$

Now, by the binomial theorem,

$$(1 - 4\lambda)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-4\lambda)^n \quad (\text{provided } |\lambda| < 1/4) = \sum_{n=0}^{\infty}\binom{2n}{n}\lambda^n.$$

Integrating with respect to λ, we see that

$$(-1/2)(1 - 4\lambda)^{1/2} = c + \sum_{n=0}^{\infty}\frac{\binom{2n}{n}}{n+1}\lambda^{n+1}, \text{ for some constant } c.$$

Setting $\lambda = 0$, we find that $c = -1/2$. Also, observe that $C_{n+1} = \dfrac{\binom{2n}{n}}{n+1}$, the $(n + 1)$th Catalan number. Therefore,

$$r = r(\lambda) = \sum_{n=0}^{\infty} C_{n+1}\lambda^n. \quad \text{Q.E.D.}$$

Also solved by the proposer.

The Big Six

H-298 *Proposed by L. Kuipers, Mollens, Valais, Switzerland.*
(Vol. 17, No. 1, February 1979)

Prove:

(i) $F_{n+1}^6 - 3F_{n+1}^5F_n + 5F_{n+1}^3F_n^3 - 3F_{n+1}F_n^5 - F_n^6 = (-1)^n, \; n = 0, 1, \ldots;$

(ii) $F_{n+6}^6 - 14F_{n+5}^6 - 90F_{n+4}^6 + 350F_{n+3}^6 - 90F_{n+2}^6 - 14F_{n+1}^6 + F_n^6$
$= (-1)^n 80, \; n = 0, 1, \ldots;$

(iii) $F_{n+6}^6 - 13F_{n+5}^6 + 41F_{n+4}^6 - 41F_{n+3}^6 + 13F_{n+2}^6 - F_{n+1}^6$
$\equiv -40 + \frac{1}{2}(1 + (-1)^n)80 \pmod{144}.$

Solution by L. Carlitz, Duke University, Durham, NC.

(i) Let $P(x, y) = x^6 - 3x^5y + 5x^3y^3 - 3xy^5 - y^6$. It is easily verified that $P(y + z, y) = -P(y, z)$.

In this identity, take $y = F_n$, $z = F_{n-1}$. This gives

$$P(F_{n+1}, F_n) = -P(F_n, F_{n-1}) \quad (n = 1, 2, 3, \ldots).$$

Thus,

$$P(F_{n+1}, F_n) = (-1)^n P(F_1, F_0) = (-1)^n P(1, 0).$$

Since $P(1, 0) = 1$, we get

$$P(F_{n+1}, F_n) = (-1)^n \quad (n = 0, 1, 2, \ldots),$$

as asserted.

(ii) Put

$$f_k(x) = \sum_{n=0}^{\infty} F_{n+1}^k x^n \quad (k = 0, 1, 2, \ldots).$$

It has been proved (L. Carlitz, "Generating Functions for Powers of Certain Sequences of Numbers," *Duke Math. Journal* 29 (1962):521-537; see also Riordan, "Generating Functions for Powers of Fibonacci Numbers," *Duke Math. Journal* 29 (1962):5-12) that

$$(*) \qquad f_k(x) = \frac{U_k(x)}{D_k(x)},$$

where

$$D_k(x) = \sum_{r=0}^{k+1} (-1)^{(1/2)r(r+1)} \frac{F_{k+1}F_k \cdots F_{k-r+2}}{F_1F_2 \cdots F_r} x^r$$

and

$$U_k(x) = \sum_{j=0}^{k-1} U_{kj} x^j \ (k = 1, 2, 3, \ldots)$$

can be computed recursively by means of

$$U_{k+1,j} = F_{j+1}U_{k,j} + (-1)^j F_{k-j+1}U_{k,j-1}.$$

For $k = 6$, we find that

$$D_6(x) = 1 - 13x - 104x^2 + 260x^3 + 260x^4 - 104x^5 - 13x^6 + x^7$$

and

$$U_6(x) = 1 - 12x - 53x^2 + 53x^3 + 12x^4 - x^5.$$

Moreover, it can be verified that

$$D_6(x) = (1 + x)(1 - 14x - 90x^2 + 350x^3 - 90x^4 - 14x^5 + x^6)$$

and

$$U_6(x) = (1 + x)(1 - 13x - 40x^2 + 93x^3 - 81x^4) + 80x^5.$$

Thus, taking $k = 6$ in (*), we have

$$(1 - 14x - 90x^2 + 350x^3 - 90x^4 - 14x^5 + x^6)\sum_{n=0}^{\infty} F_{n+1}^6 x^n = \frac{U_6(x)}{1 + x}$$

$$= 1 - 13x - 40x^2 + 93x^3 - 81x^4 + \frac{80x^5}{1 + x}$$

$$= 1 - 13x - 40x^2 + 93x^3 - 81x^4 + 80\sum_{n=0}^{\infty}(-1)^n x^{n+5}.$$

Comparing coefficients of x^{n+5}, we get

$$F_{n+6}^6 - 14F_{n+5}^6 - 90F_{n+4}^6 + 350F_{n+3}^6 - 90F_{n+2}^6 - 14F_{n+1}^6 + F_n^6$$
$$= (-1)^n 80 \quad (n = 0, 1, 2, \ldots).$$

(iii) We have

$$1 - 14x - 90x^2 + 350x^3 - 90x^4 - 14x^5 + x^6$$
$$\equiv (1 - x)(1 - 13x + 41x^2 - 41x^3 + 13x^4 - x^5) \pmod{144},$$

so that

$$D_6(x) \equiv (1 - x^2)(1 - 13x + 41x^2 - 41x^3 + 13x^4 - x^5) \pmod{144}.$$

It follows that

$$(1 - 13x + 41x^2 - 41x^3 + 13x^4 - x^5)\sum_{n=0}^{\infty} F_{n+1}^6 x^n \equiv \frac{U_6(x)}{1 - x^2}$$
$$\equiv \frac{1 - 11x - 64x^2 - 11x^3 + x^4}{1 + x}$$
$$\equiv 1 - 12x - 52x^2 + 41x^3 - \frac{40x^4}{1+x} \pmod{144}.$$

Comparing coefficients of x^{n+5}, we get

$$F_{n+6}^6 - 13F_{n+5}^6 + 41F_{n+4}^6 - 41F_{n+3}^6 + 13F_{n+2}^6 - F_{n+1}^6$$
$$\equiv (-1)^n 40 \pmod{144} \quad (n = 0, 1, 2, \ldots).$$

Also solved by P. Bruckman, G. Wulczyn, D. Zeitlin, and the proposer.

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