GENERALIZED FIBONACCI NUMBERS

ANNE SILVA
University of Santa Clara, Santa Clara, CA 95053
VERNER E. HOGGATT, JR.
San Jose State University, San Jose, CA 95192

The sequence of generalized Fibonacci numbers is composed of terms derived from Pascal's triangle. The nth term of the sequence, u_n, is equal to the sum from $i = 0$ to $i = \frac{n + p - 1}{p + 1}$ of the terms $\binom{n - (i - 1)p - 1}{i}$, which represent binomial coefficients.

In the left-justified form of Pascal's triangle, u_n equals the sum of the $\binom{n + p - 1}{0}$ term and the terms taken successively p units up and 1 unit over. For $p = 0$, this generates the powers of 2. For $p = 1$, the resulting sequence is the Fibonacci numbers.

The sequence for $p = k$, any given constant, begins as follows:

$$\begin{matrix} u_0 & u_1 & u_2 & \ldots & u_k & u_{k+1} & u_{k+2} & u_{k+3} & \ldots \\ 1 & 1 & 2 & \ldots & k & k+1 & k+2 & k+4 & \ldots \end{matrix}$$

The rest of the sequence can be generated using the recursion formula $u_n = u_{n-1} + u_{n-k-1}$.

There are four important properties related to representations of integers which apply to the generalized Fibonacci sequence.

1. Completeness—Every positive integer N can be expressed as a sum of distinct u_n terms:

$$N = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m,\ \alpha_i \in \{0, 1\}.$$

2. Zeckendorf Form—Every positive integer N has a unique representation using a minimal number of u_n terms $\alpha_i \alpha_{i+j} = 0$ for $1 \le j \le k$.

3. Second Canonical Form—In this form, any positive integer M which contains $u_1 = 1$ in its representation has this u_1 replaced by $u_0 = 1$. This form is also unique for each positive integer.

4. Lexicographic Ordering—Both the Zeckendorf and Second Canonical forms of representations are lexicographic orderings meaning that, when comparing two numbers M and N, $M > N$ iff M has the larger coefficient for u_i, where u_i is the first summand for which the representations of M and N differ, going from highest to lowest.

The set of positive integers can be partitioned into $k + 1$ sets, using representations in terms of generalized Fibonacci numbers. Since the sequence of generalized Fibonacci numbers is complete with respect to the positive integers, each positive integer N is the sum of distinct u_n terms. The partitions are made according to the subscript on the smallest u_n term used in the Zeckendorf representation of an integer. If the subscript is congruent to i modulo $(k+1)$, then that integer is an element of the set A_i. Every integer is an element of one and only one set A_i for $1 \le i \le k+1$. The notation $A_i(n)$ denotes the nth element of the set A_i, when the elements are listed in natural order.

$A_i(n) = R + u_{m(k+1)+i}$, where R denotes the representation of N minus the smallest summand.

$u_{m(k+1)+i}$ can be rewritten using the recursion formula:

$$u_n = u_{n-1} + u_{n-k-1}.$$

$$\begin{aligned} u_{m(k+1)+i} &= u_{m(k+1)+i-1} + u_{m(k+1)+i-(k+1)} = u_{m(k+1)+i-1} + u_{(m-1)(k+1)+i} \\ &= u_{m(k+1)+i-1} + u_{(m-1)(k+1)+i-1} + u_{(m-2)(k+1)+i} \end{aligned}$$

$$= u_{m(k+1)+i-1} + u_{(m-1)(k+1)+i-1} + \cdots + u_i.$$

Thus, $A_i(n) = u_{m(k+1)+i} + R$ can be rewritten

$$A_i(n) = u_i + R', \ 1 \le i \le k+1.$$

R' is the rest of the representation of $A_i(n)$, so

$$A_i(n) = u_i + \alpha_{i+1}u_{i+1} + \alpha_{i+2}u_{i+2} + \alpha_{i+3}u_{i+3} + \cdots + \alpha_m u_m.$$

There are two mappings which are useful in discovering properties of the partitioned integers. The first is f, which advances by 1 the subscripts on the summands of N when N is written in Second Canonical form. The second mapping is f^*, which performs the same function as f on N when N is written in Zeckendorf form.

$$A_i(n) = u_i + \alpha_{i+1}u_{i+1} + \alpha_{i+2}u_{i+2} + \cdots + \alpha_n u_n.$$

$$A_i(n) \xrightarrow{f} u_{i+1} + \alpha_{i+1}u_{i+2} + \alpha_{i+2}u_{i+3} + \cdots + \alpha_n u_{n+1} = A_{i+1}(n).$$

By lexicographic ordering, $A_i(n)$ is mapped by f onto the nth element of A_{i+1}, $2 \le i \le k$.

$$A_1(n) = u_0 + \alpha_2 u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m \text{ in Second Canonical form.}$$

$$A_1(n) \xrightarrow{f} u_1 + \alpha_2 u_3 + \alpha_3 u_4 + \cdots + \alpha_m u_{m+1},$$

which is an element of A_1.

$$A_{k+1}(n) = u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_m.$$

$$A_{k+1}(n) \xrightarrow{f} u_{k+2} + \alpha_{k+2}u_{k+3} + \cdots + \alpha_m u_{m+1}.$$

By the recursion formula, $u_n = u_{n-1} + u_{n-k-1}$, $u_{k+2} = u_{k+1} + u_1$.

$$A_{k+1}(n) \xrightarrow{f} u_1 + u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_{m+1},$$

which is an element of A_1.

Since $A_1, A_2, \ldots, A_{k+1}$ cover the positive integers, we have that every integer n is mapped by f onto the set $(A_1 \cup A_3 \cup A_4 \cup \ldots \cup A_{k+1})$. By lexicographic ordering, n is mapped onto the nth element of this set. Call this set H_1. Then $n \xrightarrow{f} H_1(n)$.

An element of the set H_1 is mapped forward onto

$$(A_1 \cup A_4 \cup A_5 \cup \ldots \cup A_{k+1}),$$

since each set except A_1 and A_{k+1} map forward one set and these two map onto A_1. Call this second set H_2: $n \xrightarrow{f} H_1(n) \xrightarrow{f} H_2(n)$

In general,

$$H_i(n) = (A_1 \cup A_{i+2} \cup A_{i+3} \cup \ldots \cup A_{k+1})(n) \text{ and } H_i(n) \xrightarrow{f} H_{i+1}(n)$$

for $1 \le i \le k-2$. For $i = k-2$,

$$H_{k-2}(n) \xrightarrow{f} H_{k-1}(n) = (A_1 \cup A_{k+1})(n). \qquad H_{k-1}(n) \xrightarrow{f} A_1(n).$$

$$n \xrightarrow{f} H_1(n) \xrightarrow{f} H_2(n) \xrightarrow{f} H_3(n) \xrightarrow{f} \ldots \xrightarrow{f} H_{k-1}(n) \xrightarrow{f} A_1(n).$$

Now using the f^* mapping,

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \xrightarrow{f^*} u_2 + \alpha_2 u_3 + \cdots + \alpha_m u_{m+1} = A_2(n).$$

In general,

$$A_i(n) = u_i + \alpha_{i+1}u_{i+1} + \cdots + \alpha_m u_m \xrightarrow{f^*} A_{i+1}(n)$$

$$A_{i+1}(n) = u_{i+1} + \alpha_{i+1}u_{i+2} + \cdots + \alpha_m u_{m+1}, \text{ for } 1 \le i \le k.$$

f^* and f are the same mappings except when applied to elements of A_1, the only elements whose Second Canonical and Zeckendorf forms are different.

(1) $n \xrightarrow{f} H_1(n) \xrightarrow{f} H_2(n) \xrightarrow{f} \cdots \xrightarrow{f} H_{k-1}(n) \xrightarrow{f} A_1(n) \xrightarrow{f^*} A_2(n) \xrightarrow{f^*} \cdots \xrightarrow{f^*} A_{k+1}(n).$

The mappings can be used to identify a further relationship between the H_i and A_i sets. By (1) above, n is mapped by k successive applications of f onto $A_1(n)$. Denote this

$$n \xrightarrow{f^{(k)}} A_1(n).$$

$$n = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \xrightarrow{f^{(k)}} u_1 + \alpha_2 u_{k+2} + \cdots + \alpha_m u_{k+m} = A_1(n).$$

$$A_1(n) \xrightarrow{f^*} u_2 + \alpha_2 u_{k+3} + \cdots + \alpha_m u_{k+m+1} = A_2(n).$$

$$n + A_1(n) = (u_1 + u_1) + \alpha_2(u_2 + u_{k+2}) + \cdots + \alpha_m(u_m + u_{k+m}).$$

Using the recursion formula,

$$u_n = u_{n-1} + u_{n-k-1},\ A_1(n) + n = u_2 + \alpha_2 u_{k+3} + \cdots + \alpha_m u_{k+m+1} = A_2(n).$$

By similar proofs, any element plus its image k steps forward in the scheme described in (1) equals the element one step further in scheme (1).

(2) $$A_1(n) + n = A_2(n),$$

$$A_i(n) + H_{i-1}(n) = A_{i+1}(n) \quad \text{for} \quad 2 \le i \le k.$$

Here is an example of the representations, partitions, and mappings for $k = 3$.

The sequence of u_n's for $k = 3$ begins as follows:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7 ...
1	1	2	3	4	5	7	10 ...

$$1 = u_0 \xrightarrow{f} u_1 = 1 = H_1(1)$$
$$2 = u_2 \xrightarrow{f} u_3 = 3 = H_1(2)$$
$$3 = u_3 \xrightarrow{f} u_4 = 4 = H_1(3)$$
$$4 = u_4 \xrightarrow{f} u_5 = 5 = H_1(4)$$
$$5 = u_0 + u_4 \xrightarrow{f} u_1 + u_5 = 1 + 5 = 6 = H_1(5)$$
$$6 = u_0 + u_5 \xrightarrow{f} u_1 + u_6 = 1 + 7 = 8 = H_1(6)$$
$$7 = u_2 + u_5 \xrightarrow{f} u_3 + u_6 = 3 + 7 = 10 = H_1(7)$$
$$8 = u_0 + u_6 \xrightarrow{f} u_1 + u_7 = 1 + 10 = 11 = H_1(8)$$
$$9 = u_2 + u_6 \xrightarrow{f} u_3 + u_7 = 3 + 10 = 13 = H_1(9)$$
$$10 = u_3 + u_6 \xrightarrow{f} u_4 + u_7 = 4 + 10 = 14 = H_1(10)$$

The other mappings described in scheme (1) are derived in a similar manner. The array for $k = 3$ from $1 \le n \le 10$ follows:

n	$H_1(n)$	$H_2(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$	$A_4(n)$
1	1	1	1	2	3	4
2	3	4	5	7	10	14
3	4	5	6	9	13	18
4	5	6	8	12	17	23
5	6	8	11	16	22	30
6	8	11	15	21	29	40
7	10	14	19	26	36	50

n	$H_1(n)$	$H_2(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$	$A_4(n)$
8	11	15	20	28	39	54
9	13	18	24	33	46	64
10	14	19	25	35	49	68

Examining this array, it is soon apparent that the differences between successive elements in a given set depend on which set the subscript belongs to. Thus, it is necessary to add another layer of subscripts to discuss these differences. We want to find a general description for

$$A_j(A_i(n) + 1) - A_j(A_i(n)).$$

Denote this difference, $\Delta A_j(A_i(n))$, as $\Delta g(j, i)$.

The simplest case to start with is $\Delta g(1, 2)$. The first step is to notice that by applying lexicographic ordering to mapping scheme (1), we can see that that number of integers $N \le H_1(n)$ must equal the number of elements of A_1 that are $\le A_2(n)$. The same idea applies to any two pairs of numbers an equal number of sets apart in the mapping scheme.

Since the number of integers $N \le H_1(n) = H_1(n)$, we have that $\#A_1$ elements $\le A_2(n) = H_1(n)$. Thus, the largest A_1 element $\le A_2(n)$ is $A_1(H_1(n))$.

$$A_2(n) = u_2 + \alpha_3 u_3 + \cdots + \alpha_m u_m.$$

Since $u_2 = 2$,

$$A_2(n) - 1 = 1 + \alpha_3 u_3 + \cdots + \alpha_m u_m = u_1 + \alpha_3 u_3 + \cdots + \alpha_m u_m \in A_1.$$

Since we are dealing with integers, the closest two elements can be is 1 apart. Thus $A_2(n) - 1$ is the largest element less than A_2, and since we know it is an element of A_1, it must be $A_1(H_1(n))$.

$$A_1(H_1(n)) + 1 = A_2(n).$$

The set H_1 excludes A_2 elements, so $A_1(A_2(n))$ cannot equal any $A_1(H_1)$ element. $A_1(A_2(n)) + 1 \notin A_2$.

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n.$$

Since $u_1 = 1$,

$$A_1(n) + 1 = 2 + \alpha_2 u_2 + \cdots + \alpha_n u_n.$$

We know that $A_1(A_2(n)) + 1$ does not belong to A_2. Adding 1 to $A_1(A_2(n))$ must change the representation so that u_2 is not used. Since in the Zeckendorf form and the Second Canonical form we are dealing with you cannot have terms in the representation closer than k subscripts apart, $A_1(A_2(n)) + 1$ cannot be an element of $A_3, A_4, \ldots, A_{k+1}$. By process of elimination, $A_1(A_2(n)) + 1$ is an element of A_1. By lexicographic ordering, it must be the next element after the $A_2(n)$th element.

(3) $$\Delta g(1, 2) = 1.$$

Next we want to find $\Delta g(1, i)$ for $3 \le i \le k$. We know from mapping scheme (1) that $n \xrightarrow{f} H_1(n)$. Therefore, $A_i(n) \xrightarrow{f} H_1(A_i(n))$. But we also know from the mapping scheme that $A_i(n) \xrightarrow{f} A_{i+1}(n)$ for $2 \le i \le k$, since f and f^* are the same mappings for these elements. Thus,

(4) $$H_1(A_i(n)) = A_{i+1}(n), \quad 2 \le i \le k.$$

By lexicographic ordering and mapping scheme (1), $\#A_i$ elements $\le A_{i+1}(n) = \#n$'s $\le H_1(n) = H_1(n)$.

$$A_{i+1}(n) = u_{i+1} + \alpha_{i+2} u_{i+2} + \cdots + \alpha_m u_m.$$

$u_{i+1} = i + 1$ for $0 \le i \le k$, so $A_{i+1}(n) - 1 = i + \alpha_{i+1} + \cdots + \alpha_m u_m \in A_i$.

We know that there are $H_1(n)$ elements of $A_i \leq A_{i+1}(n)$. Therefore, the largest $A_i \leq A_{i+1}(n)$ is $A_i(H_1(n))$. We know that $A_{i+1}(n) - 1 \in A_i$, and that this is the closest any 2 integers can get. Therefore,

(5) $$A_i(H_1(n)) + 1 = A_{i+1}(n), \text{ for } 1 \leq i \leq k.$$

Equation (5) can be generalized further. By lexicographic ordering and mapping scheme (1), $\#A_i$ elements $\leq A_{i+j}(n) = \#n$'s $\leq H_j(n) = H_j(n)$, for $1 \leq i \leq k$; $1 \leq j \leq k-1$; $1 \leq i+j \leq k+1$. Thus the $H_j(n)$th element of A_i is the largest one $\leq A_{i+j}(n)$.

$$A_{i+j}(n) = u_{i+j} + \alpha_{i+j+1}u_{i+j+1} + \cdots + \alpha_m u_m.$$

$u_{i+j} = i + j$ for $1 \leq i+j \leq k+1$, so

$$A_{i+j}(n) - j = i + \alpha_{i+j+1}u_{i+j+1} + \cdots + \alpha_m u_m.$$

$i = u_i$ for $1 \leq i \leq k+1$, so

$$A_{i+j}(n) - j \in A_i.$$

By mapping scheme (1), the closest any 2 elements of A_i and A_{i+j} can be is j units apart, so $A_{i+j}(n) - j$ is the largest A_i element $\leq A_{i+j}(n)$. Thus,

(6) $$A_1(H\ (n)) + j = A_{i+j}(n) \quad \text{for } 1 \leq i \leq k;\ 1 \leq j \leq k-1;\ 1 \leq i+j \leq k+1.$$

$$A_1(H_1(A_{i-1}(n))) + 1 = A_1(A_i(n)) + 1, \text{ by (4)},$$

$$A_1(H_1(A_{i-1}(n))) + 1 = A_2(A_{i-1}(n)) \text{ by (5)}.$$

Thus (a) $$A_1(A_i(n)) + 1 = A_2(A_{i-1}(n)).$$

$$A_2(H_{i-3}(n)) + i - 3 = A_{i-1}(n) \text{ by (6)}.$$

$H_{i-3} = (A_1 \cup A_{i-1} \cup A_{i-1} \cup A_i \cup \cdots \cup A_{k+1})$ by definition of H_i (see p. 291). Thus $A_{i-1}(n) \in H_{i-3}$, and $A_2(A_{i-1}(n)) + i - 3 \in A_{i-1}$, say $A_{i-1}(t)$.

$$A_2(H_{i-2}(n)) + i - 2 = A_i(n) \text{ by (6)}.$$

$H_{i-2} = (A_1 \cup A_i \cup \cdots \cup A_{k+1})$ by definition of H_i. Thus $A_{i-1}(n) \notin H_{i-2}$, and $A_2(A_{i-1}(n)) + i - 2 \notin A_i$.

$$A_2(A_{i-1}(n)) + i - 2 = A_{i-1}(t) + 1 \notin A_i.$$

$$A_{i-1}(t) = u_{i-1} + \alpha_i u_i + \cdots + \alpha_m u_m.$$

Adding 1 to this particular A_{i-1} element must change the representation so that a u_i is not used. Since, in Zeckendorf and Second Canonical form, no two summands can have subscripts closer than k units apart, $A_{i-1}(t) + 1$ cannot use any summands from u_2, u_3, ... up to u_{k+1}. This means that $A_{i-1}(t) + 1 \notin A_2$, A_3, ... up to A_{k+1}. The only remaining set is A_1.

From (a) above, $A_1(A_i(n)) + 1 = A_2(A_{i-1}(n))$. $A_2(A_{i-1}(n)) + i - 2 \in A_1$, and this must be the next A_1 element after $A_1(A_i(n))$ by lexicographic ordering. Therefore

(7) $$\Delta g(1, i) = i - 1 \quad \text{for } 3 \leq i \leq k.$$

The next case we will examine is $\Delta g(1, 1)$. Since $n \xrightarrow{f} H_1(n)$, from mapping scheme (1),

$$A_1(n) \xrightarrow{f} H_1(A_1(n)).$$

From (1), we also know that

$$A_1(n) \xrightarrow{f^*} A_2(n).$$

The only difference between the two mappings is that f maps $u_0 = 1$ onto $u_1 = 1$, while f^* maps $u_1 = 1$ onto $u_2 = 2$. Therefore, we know that

$$H_1(A_1(n)) + 1 = A_2(n).$$

But we also know that $A_1(H_1(n)) + 1 = A_2(n)$ by (5). Thus

(8) $$A_1(H_1(n)) = H_1(A_1(n)).$$

By (4), $H_1(A_i(n)) = A_{i+1}(n)$ for $2 \le i \le k$. $H_1(A_{k+1}(n))$ is the only portion of the H_1 set not identified as a particular A_i. $H_1 = (A_1 \cup A_3 \cup A_4 \cup \ldots \cup A_{k+1})$. $A_3, A_4, \ldots, A_{k+1}$ are taken by $H_1(A_i(n))$ for $2 \le i \le k$, and $A_1(H_1(n))$ is taken by $H_1(A_1(n))$. Since the elements of each set A_i do not overlap, $H_1(A_{k+1}(n))$ must be an element of the only remaining portion of H_1: $A_1(A_2)$. By lexicographic ordering, it must be the nth.

(9) $$H_1(A_{k+1}(n)) = A_1(A_2(n)).$$

$$\begin{aligned} A_1(H_1(A_1(n))) + 1 &= A_2(A_1(n)) \text{ by (5)} \\ &= A_1(A_1(H_1(n))) + 1 \text{ by (8)}. \end{aligned}$$

$$\begin{aligned} A_1(H_1(A_{k+1}(n))) + 1 &= A_2(A_{k+1}(n)) \text{ by (5)} \\ &= A_1(A_1(A_2(n))) + 1 \text{ by (9)}. \end{aligned}$$

$(A_1 \cup A_{k+1}) = H_{k-1}$, so the first line of each of the above two equations defines $A_2(H_{k-1}(n))$.

$(H_1 \cup A_2)$ = all the integers; thus, the second line of each of the above two equations defines $A_1(A_1(n)) + 1$.

Thus $A_2(H_{k-1}(n)) = A_1(A_1(n)) + 1$.

$$A_2(H_{k-1}(n)) + k - 1 = A_{k+1}(n) \text{ by (6)}.$$

So

$$A_1(A_1(n)) + k = A_{k+1}(n).$$

$$A_{k+1}(n) = u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_m.$$

$$A_{k+1}(n) + 1 = u_1 + u_{k+1} + \alpha_{k+2}u_{k+2} + \cdots + \alpha_m u_m,$$

since 1 and $k+1$ are k units apart, making the combination of u_1 and u_{k+1} acceptable under Zeckendorf form.

$A_{k+1}(n) + 1 \in A_1$, since it has a u_1 in its representation. Thus

$$A_1(A_1(n)) + k + 1 \in A_1,$$

and this must be the next A_1 by lexicographic ordering.

(10) $$\Delta g(1, 1) = k + 1.$$

Finally, we examine $\Delta g(1, k+1)$. $A_1(H_{k-1}(n)) + (k - 1) = A_k(n)$ by (6). $A_{k+1}(n) \in H_{k-1}$, so $A_1(A_{k+1}(n)) + k - 1 \in A_k$.

$A_1(A_1(n)) + k = A_2(H_{k-1}(n)) + k - 1 = A_{k+1}(n)$ from the preceding argument for $\Delta g(1, 1)$.

$A_1(A_{k+1}(n)) + k \notin A_{k+1}$, since A_1 and A_{k+1} are disjoint sets.

$A_1(A_{k+1}(n)) + k = A_k(t) + 1$. Since this $A_k(t) + 1$ is not an element of A_{k+1}, it can only be an A_1 element from the restrictions imposed by Zeckendorf form.

$A_1(A_{k+1}(n)) + k \in A_1$, and it must be the next A_1 by lexicographic ordering.

(11) $$\Delta g(i, k + 1) = k.$$

Combining equations (3), (7), (10), and (11), we have that

$$\begin{aligned} \Delta g(1, 1) &= k + 1, \\ \Delta g(1, i) &= i - 1, \text{ for } 2 \le i \le k + 1. \end{aligned}$$

Since $u_i = 1$ for $1 \le i \le k + 1$, we can restate this as

$$\Delta g(1, 1) = u_{k+1},$$

$$\Delta g(1, i) = u_{i-1}.$$

Now we will use mathematical induction to prove what $\Delta g(j, i)$ is equal to.

(12) Induction Hypothesis: $\Delta g(j, 1) = u_{k+j}$,
$\Delta g(j, i) = u_{i+j-2}$, for $2 \leq i \leq k + 1$.

These differences apply for $1 \leq j \leq k + 1$.

Equations (3), (7), (10), and (11) prove that the induction hypothesis is true for $j = 1$, establishing an induction basis.

Assume

(13) $$\Delta g(m, i) = u_{i+m-2}, \text{ for } 2 \leq i \leq k + 1.$$

(a) $$A_{m+1}(A_i(n)) = A_m(H_1(A_i(n))) + 1 \text{ by (5).}$$

$$A_m(H_1(A_i(n))) + 1 = A_m(A_{i+1}(n)) + 1 \text{ by (4).}$$

$$A_m(A_{i+1}(n)) + u_{i+m-1} = A_m(A_{i+1}(n) + 1) \text{ by assumption (13), for } 1 \leq i \leq k.$$

(b) $$A_{i+1}(n) + 1 \in H_1 \text{ for } 1 \leq i \leq k.$$

Since $A_m(H_1(n)) + 1 = A_{m+1}(n)$ by (5), $A_m(A_{i+1}(n) + 1) + 1 \in A_{m+i}$.

$$A_m(H_1(A_i(n))) + 1 = A_{m+1}(A_i(n)) \text{ from (13a)}$$
$$= A_m(A_{i+1}(n)) + 1.$$

Thus

$$A_{m+1}(A_i(n)) = A_m(A_{i+1}(n)) + 1.$$

$$A_{m+1}(A_i(n)) + u_{i+m-1} = A_m(A_{i+1}(n)) + u_{i+m-1} + 1$$
$$= A_m(A_{i+1}(n) + 1) + 1 \text{ by (13).}$$

Thus $$A_m(A_{i+1}(n) + 1) + 1 \in A_{m+1} \text{ by 13b),}$$

and by (5), $$A_{m+1}(A_i(n)) + u_{i+m-1} \in A_{m+1}.$$

This must be the next A_{m+1}, so

$$\Delta g(m + 1, i) = u_i + (m + 1) - 2 = u_{i+m-1}.$$

Thus far, assuming $\Delta g(m, i) = u_{i+m-2}$ has implied that

$$\Delta g(m + 1, i) = u_{i+(m+1)-2}, \text{ for } 2 \leq i \leq k.$$

By mathematical induction, hypothesis (12) holds true for $2 \leq i \leq k$.

Assume

(14) $$\Delta g(m, k + 1) = u_{k+m-1} \text{ and } \Delta g(m, 1) = u_{k+m}.$$

$$A_{m+1}(A_{k+1}(n)) = A_m(H_1(A_{k+1}(n))) + 1 \text{ by (5)}$$
$$= A_m(A_1(A_2(n))) + 1 \text{ by (9).}$$

$$A_m(A_1(A_2(n))) + u_{k+m} = A_m(A_1(A_2(n)) + 1) \text{ by (14)}$$
$$= A_m(A_1(A_2(n) + 1)) \text{ by (3).}$$

$$A_1(A_2(n)) + 1 \in H_1, \text{ so } A_m(A_1(A_2(n) + 1)) + 1 \in A_{m+1} \text{ by (5).}$$

Putting the last few statements together,

$$A_{m+1}(A_{k+1}(n)) + u_{k+m} \in A_{m+1}.$$

This must be the next A_{m+1} element by lexicographic ordering.

$$\Delta g(m, k + 1) = u_{k+m-1} \text{ implies } \Delta g(m + 1, k + 1) = u_{k+m} = u_{k+(m+1)-1}.$$

Since $\Delta g(j, k + 1) = u_{k+j-1}$ was proved true for $j = 1$ in (11), and assuming this statement true for $j = m$ implies that it holds for $j = m + 1$, then, by mathematical induction $\Delta g(j, k + 1) = u_{k+j-1}$.

For the final case, we want to prove that $\Delta g(m, 1) = u_{k+m}$ implies that $\Delta g(m + 1, 1) = u_{k+m+1}$.

$$A_{m+1}(A_1(n)) = A_m(H_1(A_1(n))) + 1 \text{ by (5)} = A_m(A_1(H_1(n))) + 1 \text{ by (8)},$$
$$A_m(A_1(H_1(n))) + u_{k+m} = A_m(A_1(H_1(n)) + 1) \text{ by (14)} = A_m(A_2(n)) \text{ by (5)},$$
$$A_m(A_2(n)) + u_m = A_m(A_2(n) + 1) \text{ by (13)}.$$

Since $A_m(H_1(n)) + 1 = A_{m+1}(n)$ by (5), and $A_2(n) + 1 \in H_1$, then

$$A_m(A_2(n) + 1) + 1 \in A_{m+1}.$$

Combining the above statements,

$$A_{m+1}(A_1(n)) + u_{k+m} + u_m \in A_{m+1}.$$

This must be the next A_{m+1} element by lexicographic ordering.

$$\Delta g(m + 1, 1) = u_{k+m} + u_m.$$

By the recursion formula, $u_n = u_{n-1} + u_{n-k-1}$, $u_{k+m} + u_m = u_{k+m+1}$, so

$$\Delta g(m + 1, 1) = u_{k+m+1}.$$

By mathematical induction, hypothesis (12) has been proved true.

$$\Delta g(j, i) = u_{i+j-2},\ 2 \leq i \leq k + 1,$$
$$\Delta g(j, 1) = u_{k+j},$$

for $1 \leq j \leq k + 1$.

Arrays for k = 1, 2, and 4 follow to help illustrate the difference formula $\Delta g(i, j)$. The array for k = 3 can be found on pages 291-292 above.

<u>$k = 1$</u>: The sequence of u_n's generated for $k = 1$ in Fibonacci numbers is:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8 ...
1	1	2	3	5	8	13	21	34 ...

n	$A_1(n)$	$A_2(n)$
1	1	2
2	3	5
3	4	7
4	6	10
5	8	13
6	9	15
7	11	18
8	12	20
9	14	23
10	16	26

<u>$k = 2$</u>:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7 ...
1	1	2	3	4	6	9	13 ...

n	$H_1(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$
1	1	1	2	3
2	3	4	6	9
3	4	5	8	12
4	5	7	11	16
5	7	10	15	22
6	9	13	19	28
7	10	14	21	31
8	12	17	25	37
9	13	18	27	40
10	14	20	30	44

$k = 4$:

u_0	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9 ...
1	1	2	3	4	5	6	8	11	15 ...

n	$H_1(n)$	$H_2(n)$	$H_3(n)$	$A_1(n)$	$A_2(n)$	$A_3(n)$	$A_4(n)$	$A_5(n)$
1	1	1	1	1	2	3	4	5
2	3	4	5	6	8	11	15	20
3	4	5	6	7	10	14	19	25
4	5	6	7	9	13	18	24	31
5	6	7	9	12	17	23	30	39
6	7	9	12	16	22	29	38	50
7	9	12	16	21	28	37	49	65
8	11	15	20	26	34	45	60	80
9	12	16	21	27	36	48	64	85
10	14	19	25	32	42	56	75	100

Another question suggested by these arrays is: How many elements of a set A_j are less than a given n? To find the answer, we need a function that increments only when it passes an A_j element. This function turns out to be the third difference of terms in successive A_j sets.

(15) $\#A_j\text{'s} < n = S(j, n) = A_{k+5-j}(n) - 3A_{k+4-j}(n) + 3A_{k+3-j}(n) - A_{k+2-j}(n).$

Proof: First, we need to define the sets A_{k+2}, A_{k+3}, and A_{k+4} and show that their properties are consistent with those of $A_1, A_2, \ldots, A_{k+1}$.

$$A_{k+1}(n) \xrightarrow{f} H_1(A_{k+1}(n)) \text{ by mapping scheme (1).}$$

$$H_1(A_{k+1}(n)) = A_1(A_2(n)) \text{ by equation (9).}$$

$$A_{k+1}(n) \xrightarrow{f^*} A_1(A_2(n)) \xrightarrow{f^*} A_2(A_2(n)) \xrightarrow{f^*} A_3(A_2(n)), \text{ using (1)}$$

with the subscript $A_2(n)$ instead of n.

$A_1(n)$ is mapped onto $A_{k+1}(n)$ by k applications of f^*.

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m \xrightarrow{f^{*(k)}} u_{k+1} + \alpha_2 u_{k+2} + \cdots + \alpha_m u_{k+m} = A_{k+1}(n).$$

$$A_{k+1}(n) \xrightarrow{f^*} u_{k+2} + \alpha_2 u_{k+3} + \cdots + \alpha_m u_{k+m+1} = u_1 + u_{k+1} + \alpha_2 u_{k+3} + \cdots = A_1(A_2(n)).$$

$$A_1(n) = u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$$

$$A_{k+1}(n) = u_{k+1} + \alpha_2 u_{k+2} + \cdots + \alpha_m u_{k+m}$$

$$A_1(n) + A_{k+1}(n) = u_1 + u_{k+1} + \alpha_2(u_2 + u_{k+2}) + \cdots + \alpha_m(u_m + u_{k+m})$$

$$= u_1 + u_{k+1} + \alpha_2 u_{k+3} + \cdots + \alpha\, u_{k+m+1},$$

by the recursion formula.

$A_1(n) + A_{k+1}(n) = A_1(A_2(n))$. Relabel $A_1(A_2(n))$ as $A_{k+2}(n)$.

Since $A_2(n)$ and $A_{k+2}(n)$ are also k applications of f^* apart in the mapping scheme, $A_2(n) + A_{k+2}(n) = A_2(A_2(n))$ by the recursion formula, since

$$A_{k+2}(n) \xrightarrow{f^*} A_2(A_2(n)).$$

Relabel $A_2(A_2(n)) = A_{k+3}(n)$.

Similarly, $A_3(n) + A_{k+3}(n) = A_3(A_2(n)) = A_{k+4}(n)$.

$$\Delta g(k+2, i) = \Delta g(1, i) + \Delta g(k+1, i) = u_{i-1} + u_{k+i-1} = u_{k+i}, \quad \text{for } 2 \le i \le k+1.$$

$$\Delta g(k+2, 1) = \Delta g(1, 1) + \Delta g(k+1, 1) = u_{k+1} + u_{2k+1} = u_{2k+2}.$$

This result is consistent with formula (12) above.

$\Delta g(k+3, i) = \Delta g(2, i) + \Delta g(k+2, i) = u_i + u_{k+i} = u_{k+i+1}$ for $2 \le i \le k+1$.

$\Delta g(k+3, 1) = \Delta g(2, 1) + \Delta g(k+2, 1) = u_{k+2} + u_{2k+2} = u_{2k+3}$.

$\Delta g(k+4, i) = \Delta g(3, i) + \Delta g(k+3, i) = u_{i+1} + u_{k+i+1} = u_{k+i+2}$ for $2 \le i \le k+1$.

$\Delta g(k+4, 1) = \Delta g(3, 1) + \Delta g(k+3, 1) = u_{k+3} + u_{2k+3} = u_{2k+4}$.

Thus, equation (12) can be extended to cover $1 \le j \le k+4$:

$$\Delta g(j, i) = u_{i+j-2},\ 2 \le i \le k+1,$$
$$\Delta g(j, 1) = u_{k+j}.$$

Now function (15) is defined for all values of j and i.

$S(j, n) = \#A_j$'s $< n = A_{k+5-j}(n) - 3A_{k+4-j}(n) + 3A_{k+3-j}(n) - A_{k+2-j}(n)$ by (15).

To prove that $S(j, n)$ increments only when passing an A_j element, look at

$$S(j, n+1) - S(j, n) = \Delta S(j, n).$$

$$\Delta S(j, n) = u_{k+3-j+i} - 3u_{k+2-j+i} + 3u_{k+1-j+i} - u_{k-j+i} \text{ for } 2 \le i \le k+1.$$

$$\Delta S(j, n) = (u_{k+3-j+i} - u_{k+2-j+i}) - 2(u_{k+2-j+i} - u_{k+1-j+i}) + (u_{k+1-j+i} - u_{k-j+i}).$$

Using the recursion formula, $\Delta S(j, n)$ reduces to

$$u_{2-j+i} - 2u_{1-j+i} + u_{-j+i} = (u_{2+i-j} - u_{1+i-j}) - (u_{1+i-j} - u_{i-j}).$$

Looking at the series of u_n's, we find that the only time this function $= 1$ is when $i = j$, so $\Delta S(j, n) = (u_2 - u_1) - (u_1 - u_0) = 1 - 0 = 1$.

This happens because $u_i = i$ for $1 \le i \le k+1$, and because, by the recursion formula, $u_{-i} = 1$ for $-k < -i \le 0$. Any other successive difference of 3 consecutive u_i terms equals $1 - 1 = 0$ for $i > j$ or $0 - 0 = 0$ for $i < j$.

Thus, $S(j, n)$ increments 1 iff $n \in A_j$ for $2 \le i \le k+1$. Since $i = 1$ has a distinct difference, that case has to be proved separately.

$$\begin{aligned}\Delta S(j, n) &= u_{2k+5-j} - 3u_{2k+4-j} + 3u_{2k+3-j} - u_{2k+2-j}\\ &= (u_{2k+5-j} - u_{2k+4-j}) - 2(u_{2k+4-j} + u_{2k+3-j})\\ &\qquad + (u_{2k+3-j} - u_{2k+2-j})\\ &= u_{k+4-j} - 2u_{k+3-j} + u_{k+2-j}\\ &= (u_{k+4-j} - u_{k+3-j}) - (u_{k+3-j} - u_{k+2-j})\\ &= u_{3-j} - u_{2-j}\\ &= 0 \text{ except for } j = 1, \text{ when it equals } 1.\end{aligned}$$

Thus, $S(j, n)$ increments 1 for $i = 1$ only when j also $= 1$.

The function $S(j, n)$ has been proved to be accurate to within a constant by examining $\Delta S(j, n)$. If a constant were present at the end of the function, it would cancel out in the incrementation process. To find out the value of the constant, it is necessary to check $S(j, 1)$ for $1 \le j \le k+1$.

$$A_1(1) = 1,\ A_2(1) = 2,\ \ldots,\ A_{k+1}(1) = k+1,$$
$$A_1(A_2(1)) = A_{k+2}(1) = A_1(2) = k+2,$$
$$A_2(A_2(1)) = A_{k+3}(1) = A_2(2) = k+4,$$
$$A_3(A_2(1)) = A_{k+4}(1) = A_3(2) = k+7.$$

These values were derived from the difference formula (12) above.

$$\begin{aligned}S(1, 1) &= A_{k+4}(1) - 3A_{k+3}(1) + 3A_{k+2}(1) - A_{k+1}(1)\\ &= (k+7) - 3(k+4) + 3(k+2) - (k+1) = 0,\end{aligned}$$

$$S(2,\ 1) = A_{k+3}(1) - 3A_{k+2}(1) + 3A_{k+1}(1) - A_k(1)$$
$$= (k + 4) - 3(k + 2) + 3(k + 1) - k = 1,$$
$$S(3,\ 1) = A_{k+2}(1) - 3A_{k+1}(1) + 3A_k(1) - A_{k-1}(1)$$
$$= (k + 2) - 3(k + 1) + 3(k) - (k - 1) = 0,$$
$$S(j,\ 1) = A_{k+5-j} - 3(1)A_{k+4-j}(1) + 3A_{k+3-j}(1) = A_{k+2-j}(1)$$
$$= (k + 5 - j) - (k + 4 - j) + (k + 3 - j) - (k + 2 - j)$$
$$= 0 \text{ for } 4 \le j \le k + 1.$$

Finally,
$$S(1,\ n) = A_{k+4}(n) - 3A_{k+3}(n) + 3A_{k+2}(n) - A_{k+1}(n),$$
$$S(2,\ n) = A_{k+3}(n) - 3A_{k+2}(n) + 3A_{k+1}(n) - A_k(n) - 1,$$
$$S(j,\ n) = A_{k+5-j}(n) - 3A_{k+4-j}(n) + 3A_{k+3-j}(n) - A_{k+2-j},$$
$$\text{for } 3 \le j \le k + 1.$$

REFERENCE

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A NOTE ON TAKE-AWAY GAMES*

ROBERT J. EPP AND THOMAS S. FERGUSON
Department of Mathematics, UCLA, Los Angeles, CA 90024

1. SUMMARY

Schwenk [1] considers take-away games where the players alternately remove a positive number of counters from a single pile, the player removing the last counter being the winner. On his initial move, the player moving first can remove at most a given number m of counters. On each subsequent move, a player can remove at most $f(n)$ counters, where n is the number of counters removed by his opponent on the preceding move. In [1], Schwenk solves the case when $f(n)$ is nondecreasing and $f(n) \ge n$. This solution is extended to the case when $f(n)$ is nondecreasing and $f(1) \ge 1$.

2. THE WINNING REPRESENTATION

Let $f(n) \ge 1$ be a nondecreasing function defining a take-away game. If a player whose turn it is to move is confronted with a pile of $n \ge 1$ counters, let $L(n)$ be the minimal number of counters he must remove in order to assure a win. Let $L(0) = \infty$. Note that $L(n) \le n$ for $n \ge 1$ and that equality might hold. Note also that removing k counters from a pile of n is a winning strategy if and only if $f(k) < L(n - k)$.

Theorem 2.1: Suppose $f(k) < L(n - k)$; then $k = L(n)$ if and only if $L(k) = k$.

Proof: Suppose that $L(k) < k$. By removing $L(k)$ counters from a pile of counters, a player can then guarantee he will eventually remove the last of the first k counters, and that he will do this by removing $\ell < k$ counters. His opponent will than face a pile of $n - k$ counters and be able to remove at

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