ADVANCED PROBLEMS AND SOLUTIONS

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-322 *Proposed by Andreas N. Philippou, American Univ. of Beirut, Lebanon*

For each fixed integer $k \geq 2$, define the k-Fibonacci sequence $f_n^{(k)}$ by

$$f_0^{(k)} = 0,\ f_1^{(k)} = 1, \text{ and}$$

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_0^{(k)} & \text{if } 2 \leq n \leq k \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \geq k+1. \end{cases}$$

Show the following:

(a) $f_n^{(k)} = 2^{n-2}$ if $2 \leq n \leq k+1$;

(b) $f_n^{(k)} < 2^{n-2}$ if $n \geq k+2$;

(c) $\sum_{n=1}^{\infty} (f^{(k)}/2^n) = 2^{k-1}$.

H-323 *Proposed by Paul Bruckman, Concord, CA*

Let $(x_n)_0^\infty$ and $(y_n)_0^\infty$ be two sequences satisfying the common recurrence

$$p(E)z_n = 0, \tag{1}$$

where p is a monic polynomial of degree 2 and $E = 1 + \Delta$ is the unit right-shift operator of finite difference theory. Show that

$$x_n y_{n+1} - x_{n+1} y_n = (p(0))^n (x_0 y_1 - x_1 y_0),\ n = 0, 1, 2, \ldots . \tag{2}$$

Generalize to the case where p is of degree $e \geq 1$.

H-324 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA*

Establish the identity

$$\begin{aligned} A \equiv\ & F_{14r}(F_{n+14r}^7 + F_n^7) - 7F_{10r}(F_{n+4r}^6 F_n + F_{n+4r} F_n^6) \\ & + 21F_{6r}(F_{n+4r}^5 F_n^2 + F_{n+4r}^2 F_n^5) - 35F_{2r}(F_{n+4r}^4 F_n^3 + F_{n+4r}^3 F_n^4) \\ =\ & F_{4r}^7 F_{7n+14}. \end{aligned}$$

H-325 *Proposed by Leonard Carlitz, Duke University, Durham, NC*

For arbitrary a, b put

$$S_m(a, b) = \sum_{j+k=m} \binom{a}{j}\binom{b+k-1}{k} \quad (m = 0, 1, 2, \ldots).$$

Show that

$$\sum_{m+n=p} S_m(a,\ b)S_n(c,\ d) = S_p(a + c,\ b + d) \tag{1}$$

$$\sum_{m+n=p} (-1)^n S_m(a,\ b)S_n(c,\ d) = S_p(a - d,\ b - c). \tag{2}$$

H-326 *Proposed by Larry Taylor, Briarwood, NY*

(A) If $p \equiv 7$ or 31 (mod 36) is prime and $(p - 1)/6$ is also prime, prove that $32(1 \pm \sqrt{-3})$ is a primitive root of p.

(B) If $p \equiv 13$ or 61 (mod 72) is prime and $(p - 1)/12$ is also prime, prove that $32(\sqrt{-1}) \pm \sqrt{3})$ is a primitive root of p.

For example:

$11 \equiv \sqrt{-3}$ (mod 31), 12 and 21 are primitive roots of 31;

$11 \equiv \sqrt{-1}$ (mod 61), $8 \equiv \sqrt{3}$ (mod 61), 59 and 35 are primitive roots of 61.

SOLUTIONS

Vandermonde

H-299 *Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA (Vol. 17, No. 2, April 1979)*

(A)

$$\text{Evaluate } \Delta = \begin{vmatrix} F_{2r} & F_{6r} & F_{10r} & F_{14r} & F_{18r} \\ F_{4r} & F_{12r} & F_{20r} & F_{28r} & F_{36r} \\ F_{6r} & F_{18r} & F_{30r} & F_{42r} & F_{54r} \\ F_{8r} & F_{28r} & F_{40r} & F_{56r} & F_{72r} \\ F_{10r} & F_{30r} & F_{50r} & F_{70r} & F_{90r} \end{vmatrix}$$

(B)

$$\text{Evaluate } D = \begin{vmatrix} 1 & L_{2r+1} & L_{4r+2} & L_{6r+3} & L_{8r+4} \\ 1 & -L_{6r+3} & L_{12r+6} & L_{18r+9} & L_{24r+12} \\ 1 & L_{10r+5} & L_{20r+10} & L_{30r+15} & L_{40r+20} \\ 1 & -L_{14r+7} & L_{28r+14} & -L_{42r+21} & L_{56r+28} \\ 1 & L_{18r+9} & L_{36r+18} & L_{54r+27} & L_{72r+36} \end{vmatrix}$$

(C)

$$\text{Evaluate } D_1 = \begin{vmatrix} 1 & L_{2r} & L_{4r} & L_{6r} & L_{8r} \\ 1 & L_{6r} & L_{12r} & L_{18r} & L_{24r} \\ 1 & L_{10r} & L_{20r} & L_{30r} & L_{40r} \\ 1 & L_{18r} & L_{36r} & L_{54r} & L_{72r} \end{vmatrix}$$

Solution by the proposer

(A) Taking out the common column factors

$$F_{2r},\ F_{6r},\ F_{10r},\ F_{14r},\ \text{and } F_{18r}$$

and simplifying, we obtain:

$$\Delta = F_{2r}F_{6r}F_{10r}F_{14r}F_{18r}\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ L_{2r} & L_{6r} & L_{10r} & L_{14r} & L_{18r} \\ L_{4r} & L_{12r} & L_{20r} & L_{28r} & L_{36r} \\ L_{6r} & L_{18r} & L_{30r} & L_{42r} & L_{56r} \\ L_{8r} & L_{24r} & L_{40r} & L_{56r} & L_{72r} \end{vmatrix}$$

$$= F_{2r}F_{6r}F_{10r}F_{14r}F_{18r}(L_{6r} - L_{2r})(L_{10r} - L_{2r})(L_{14r} - L_{2r})(L_{18r} - L_{2r})$$
$$(L_{10r} - L_{6r})(L_{14r} - L_{6r})(L_{18r} - L_{6r})$$
$$(L_{14r} - L_{10r})(L_{18r} - L_{10r})$$
$$(L_{18r} - L_{14r})$$
$$= 5^{10}F_{2r}F_{6r}F_{10r}F_{14r}F_{18r}F_{2r}^4F_{4r}^4F_{6r}^3F_{8r}^3F_{10r}^2F_{12r}^2F_{14r}F_{16r}$$
$$= 5^{10}F_{2r}^5F_{4r}^4F_{6r}^4F_{8r}^3F_{10r}^3F_{12r}^2F_{14r}^2F_{16r}F_{18r}.$$

(B) The solution is as follows:

(1) $L_{6r+3} + L_{2r+1} = 5F_{4r+2}\,F_{2r+1}$ ... (5) $L_{14r+7} - L_{6r+3} = 5F_{10r+5}\,F_{4r+2}$
(2) $L_{12r+6} + L_{4r+2} = 5F_{8r+4}\,F_{4r+2}$ ... (6) $L_{28r+14} - L_{12r+6} = 5F_{20r+10}F_{8r+4}$
(3) $L_{18r+9} + L_{6r+3} = 5F_{12r+6}F_{6r+3}$ ... (7) $L_{42r+21} - L_{18r+9} = 5F_{30r+15}F_{12r+6}$
(4) $L_{24r+12} + L_{8r+4} = 5F_{16r+8}F_{8r+4}$ ... (8) $L_{56r+27} - L_{24r+12} = 5F_{40r+20}F_{16r+8}$

(1) divides (2), (3), (4), ... (5) divides (6), (7), (8).

$$D = (L_{6r+3} + L_{2r+1})(L_{10r+5} - L_{2r+1})(L_{14r+7} + L_{2r+1})(L_{18r+9} - L_{2r+1})$$
$$(L_{10r+5} + L_{6r+3})(L_{14r+7} - L_{6r+3})(L_{18r+9} + L_{6r+3})$$
$$(L_{14r+7} + L_{10r+5})(L_{18r+9} - L_{10r+5})$$
$$(L_{18r+9} + L_{14r+7})$$
$$= 5^{10}F_{2r+1}^4F_{4r+2}^4F_{6r+3}^3F_{8r+4}^3F_{10r+5}^2F_{12r+6}^2F_{14r+7}F_{16r+8}.$$

(C) The solution is as follows:

(1) $L_{r(4t+2)} - L_{r(4s+2)} = 5F_{r(2s+2t+2)}F_{r(2t-2s)}$ (3)
(2) $L_{rk(4t+2)} - L_{rk(4s+2)} = 5F_{rk(2s+2t+2)}F_{rk(2t-2s)}$ (4)

Since (3) divides (4), (1) divides (2). Checking for proper degree and sign, the sum of the subscripts in the main diagonal, we have

$$D_1 = (L_{6r} - L_{2r})(L_{10r} - L_{2r})(L_{14r} - L_{2r})(L_{18r} - L_{2r})$$
$$(L_{10r} - L_{6r})(L_{14r} - L_{6r})(L_{18r} - L_{6r})$$
$$(L_{14r} - L_{10r})(L_{18r} - L_{10r})$$
$$(L_{18r} - L_{14r})$$

or $D_1 = 5^{10}F_{2r}^4F_{4r}^4F_{6r}^3F_{8r}^3F_{10r}^2F_{12r}^2F_{14r}F_{16r}$.

Sum Difference

H-301 *Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, CA (Vol. 17, No. 2, April 1979)*

Let $A_0, A_1, A_2, \ldots, A_n, \ldots$ be a sequence such that the nth differences are zero (that is, the diagonal sequence terminates). Show that, if

$$A(x) = \sum_{i=0}^{\infty} A_i x^i,$$

then

$$A(x) = 1/(1 - x)D(x/(1 - x)),$$

where

$$D(x) = \sum_{i=0}^{\infty} d_i x^i.$$

Solution by Paul Bruckman, Concord, CA

It is assumed that the d_i's, which are not explicitly defined, are in fact, defined as $d_i \equiv \Delta^i A_0$. Then,

$$\frac{1}{(1-x)} D\left(\frac{x}{(1-x)}\right) = \sum_{i=0}^{\infty} d_i x^i (1 - x)^{-i-1} = \sum_{i=0}^{\infty} d_i x^i \sum_{k=0}^{\infty} \binom{-i-1}{k} (-x)^k$$

$$= \sum_{i=0}^{\infty} d_i \sum_{k=0}^{\infty} \binom{i+k}{i} x^{i+k} = \sum_{i=0}^{\infty} d_i \sum_{k=i}^{\infty} \binom{k}{i} x^k$$

$$= \sum_{k=0}^{\infty} x^k \sum_{i=0}^{k} \binom{k}{i} d_i = \sum_{k=0}^{\infty} x^k \sum_{i=0}^{k} \binom{k}{i} \Delta^i A_0$$

$$= \sum_{k=0}^{\infty} x^k (1 + \Delta)^k A_0 = \sum_{k=0}^{\infty} x^k E^k A_0$$

$$= \sum_{k=0}^{\infty} A_k x^k = A(x). \quad \text{Q.E.D.}$$

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