Therefore, NEWTON(x_{n-1}) = SECANT(x_{n-1}, x_{n-1}), and so (iii) follows from (iv). Note that this identity holds for any polynomial equation $f(x) = 0$.

(iv) By (6),

$$\begin{aligned}\text{SECANT}(u_{m+1}/u_m,\ u_{n+1}/u_n) &= \frac{a(u_{m+1}/u_m)(u_{n+1}/u_n) - c}{a(u_{m+1}/u_m + u_{n+1}/u_n) + b}\\ &= \frac{au_{m+1}u_{n+1} - cu_m u_n}{au_{m+1}u_n + au_m u_{n+1} + bu_m u_n}\\ &= \frac{au_{m+1}u_{n+1} - cu_m u_n}{au_{m+1}u_n - cu_m u_{n-1}}\\ &= au_{m+n+1}/au_{m+n} \quad \text{(by the lemma)}\\ &= u_{m+n+1}/u_{m+n}. \ \square\end{aligned}$$

Remarks:

1. The theorem does not generalize to polynomials of degree higher than 2.
2. Not only do the ratios of the consecutive Fibonacci numbers converge to φ, they are the "best" rational approximation to φ; i.e., if $n > 1$, $0 < F \leq F_n$ and $P/F \neq F_{n+1}/F_n$, then $|F_{n+1}/F_n - \varphi| < |P/F - \varphi|$ by [4]. Since Newton's method and the secant method produce subsequences of Fibonacci ratios, they also produce the best rational approximation to φ.

ACKNOWLEDGMENTS

The above results were discovered in response to a question posed by Ernst Specker. This research was supported by the National Science Foundation under Grant GK-43121 at Stanford University and by the Forschungsinstitut für Mathematik, Eidgenössische Technische Hochschule, Zürich.

REFERENCES

1. P. Henrici. *Elements of Numerical Analysis*. New York: Wiley, 1964.
2. D. E. Knuth. *The Art of Computer Programming*. Vol. I: *Fundamental Algorithms*. Reading, Mass.: Addison-Wesley, 1968.
3. A. Ralston. *A First Course in Numerical Analysis*. New York: McGraw-Hill, 1965.
4. G. H. Hardy & E. M. Wright. *An Introduction to the Theory of Numbers*, p. 151, Theorem 181. London: Oxford, 1968.

*****

A CHARACTERIZATION OF THE FUNDAMENTAL SOLUTIONS TO PELL'S EQUATION $u^2 - Dv^2 = C$

M. J. DeLEON
Florida Atlantic University, Boca Raton, FL 33432

Due to a confusion originating with Euler, the diophantine equation

(1) $$u^2 - Dv^2 = C,$$

where D is a positive integer that is not a perfect square and C is a nonzero integer, is usually called *Pell's equation*. In a previous article [1, Theorem 2], the following theorem was proved.

Theorem 1: Let $x_1 + y_1\sqrt{D}$ be the fundamental solution to $x^2 - Dy^2 = 1$. If $k =$

$(y_1)/(x_1 - 1)$ and if $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = -N$, where $N > 0$, then $v_0 = |v_0| \geq k|u_0|$. If $k = (Dy_1)/(x_1 - 1)$ and if $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = N$, where $N > 1$, then $u_0 = |u_0| \geq k|v_0|$.

In Theorem 4, we shall prove the converse of this result. In the sequel, the definition of a fundamental solution to Eq. (1) given in [1] will be used. This definition differs from the one in [2, p. 205] only when $v_0 < 0$. In this case, if the fundamental solution given in [1] is denoted by $u_0 + v_0\sqrt{D}$, then the one given by the definition in [2] would be $-(u_0 + v_0\sqrt{D})$. We shall need to recall Remark A of [1] and to add to the three statements of this remark the statement:

(iv) If $C \leq 1$ and $-u_0 + v_0\sqrt{D}$ is in K then $u_0 \geq 0$. If $C \geq 1$ and $u_0 - v_0\sqrt{D}$ is in K then $v_0 \geq 0$.

Also, we shall need the following result (see [1, Theorem 5]).

Theorem 2: If $u + v\sqrt{D}$ is a solution in nonnegative integers to the diophantine equation $u^2 - Dv^2 = C$, where $C \neq 1$, then there exists a nonnegative integer n such that $u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^n$ where $u_0 + v_0\sqrt{D}$ is the fundamental solution to the class of solutions of $u^2 - Dv^2 = C$ to which $u + v\sqrt{D}$ belongs and $x_1 + y_1\sqrt{D}$ is the fundamental solution to $x^2 - Dy^2 = 1$.

We now need to prove a lemma and a simple consequence of this lemma.

Lemma 3: Let $u_0 + v_0\sqrt{D}$ be a fundamental solution to a class of solutions to $u^2 - Dv^2 = C$. If, for $n \geq 1$, we let $u_n + v_n\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^n$, then $u_n > 0$ and $v_n > 0$ for $n \geq 1$.

Proof: Since

$$u_1 + v_1\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D}) = (u_0x_1 + Dv_0y_1) + (u_0y_1 + v_0x_1)\sqrt{D},$$

we have that $u_1 = u_0x_1 + Dv_0y_1$ and $v_1 = u_0y_1 + v_0x_1$.

We now begin an induction proof of Lemma 3. First, suppose $u_0^2 - Dv_0^2 = C$, where $C < 0$. This implies, by Remark A [1], $v_0 > 0$. Hence $u_0 \geq 0$ implies $u_1 > u_0x_1 \geq u_0 \geq 0$ and $v_1 > v_0 > 0$. Thus suppose $u_0 < 0$. By Theorem 1,

$$v_0 \geq \frac{-u_0y_1}{x_1 - 1} = \frac{-u_0(x_1 + 1)}{Dy_1}.$$

Whence, $u_1 = u_0x_1 + Dv_0y_1 \geq -u_0 > 0$ and $v_1 = u_0y_1 + v_0x_1 \geq v_0 > 0$. Therefore, for $C < 0$, $u_1 > 0$ and $v_1 > 0$.

Next, suppose $u_0^2 - Dv_0^2 = C$, where $C > 0$. This implies $u_0 > 0$. Thus $v_0 \geq 0$ implies $u_1 > u_0 > 0$ and $v_1 > v_0 \geq 0$. Thus suppose $v_0 < 0$. Hence $C > 1$, so by Theorem 1,

$$u_0 \geq \frac{-Dv_0y_1}{x_1 - 1} = \frac{-v_0(x_1 + 1)}{y_1}.$$

Whence, $u_1 \geq u_0 > 0$ and $v_1 \geq -v_0 > 0$. This completes the proof of Lemma 3 for $n = 1$.

Since

$$(2) \qquad (u_{n+1} + v_{n+1}\sqrt{D}) = (u_n + v_n\sqrt{D})(x_1 + y_1\sqrt{D}) = (u_nx_1 + Dv_ny_1) + (x_1v_n + y_1u_n)\sqrt{D},$$

the assumption $u_n > 0$ and $v_n > 0$ implies $u_{n+1} > 0$ and $v_{n+1} > 0$.

Corollary: With u_0, v_0, u_n, and v_n defined as in Lemma 3, we have $u_{n+1} > u_n$ and $v_{n+1} > v_n$ for $n \geq 0$.

Proof: In the proof of Lemma 3, it was shown that $v_1 \geq v_0$ and that, in addition, for $u_0 \geq 0$ or $C > 0$ we actually have $v_1 > v_0$. For the case $u_0 < 0$ and $C < 0$, it follows from the proof of Lemma 3 that $v_1 = v_0$ implies $u_1 = -u_0$. So

$-u_0 + v_0\sqrt{D} = u_1 + v_1\sqrt{D}$ belongs to the same class of solutions to $u^2 - Dv^2 = C$ as $u_0 + v_0\sqrt{D}$. Since we are assuming $u_0 < 0$, this contradicts (iv) of Remark A [1]. Hence, even in this case, $v_1 > v_0$. In a similar manner, it is seen that we always have $u_1 > u_0$. Since $u_n > 0$ and $v_n > 0$ for $n \geq 1$, (2) implies that $u_{n+1} > u_n$ and $v_{n+1} > v_n$ for $n \geq 1$.

Theorem 4: If $u + v\sqrt{D}$ is a solution in nonnegative integers to $u^2 - Dv^2 = -N$, where $N \geq 1$, and if $v \geq ku$, where $k = (y_1)/(x_1 - 1)$, then $u + v\sqrt{D}$ is the fundamental solution of a class of solutions to $u^2 - Dv^2 = -N$. If $u + v\sqrt{D}$ is a solution in nonnegative integers to $u^2 - Dv^2 = N$, where $N > 1$, and if $u \geq kv$, where $k = (Dy_1)/(x_1 - 1)$, then $u + v\sqrt{D}$ is the fundamental solution of a class of solutions to $u^2 - Dv^2 = N$.

Proof: By Theorem 2, $u + v\sqrt{D} = (u_0 + v_0\sqrt{D})(x_1 + y_1\sqrt{D})^n = u_n + v_n\sqrt{D}$, where n is a nonnegative integer and $u_0 + v_0\sqrt{D}$ is a fundamental solution to $u^2 - Dv^2 = \pm N$. We shall prove $u + v\sqrt{D} = u_0 + v_0\sqrt{D}$. So assume $n \geq 1$. Then we have

$$\begin{aligned} u_n + v_n\sqrt{D} &= (u_{n-1} + v_{n-1}\sqrt{D})(x_1 + y_1\sqrt{D}) \\ &= (x_1u_{n-1} + Dy_1v_{n-1}) + (x_1v_{n-1} + y_1u_{n-1})\sqrt{D}. \end{aligned}$$

Thus $u_{n-1} = x_1u_n - Dy_1v_n$ and $v_{n-1} = -y_1u_n + x_1v_n$.

First, suppose $u + v\sqrt{D}$ is a solution to $u^2 - Dv^2 = -N$. We know that

$$v = v_n \geq ku_n = \frac{y_1u_n}{x_1 - 1}.$$

Hence

$$v_{n-1} = -y_1u_n + x_1v_n = (x_1 - 1)v_n - y_1u_n + v_n \geq v_n.$$

But by the corollary to Lemma 3, $v_{n-1} < v_n$ for $n \geq 1$. Thus $n = 0$ and the proof is complete for the case $u^2 - Dv^2 = -N$.

Now, suppose $u + v\sqrt{D}$ is a solution to $u^2 - Dv^2 = N$. We know that

$$u_n \geq kv_n = \frac{Dy_1v_n}{x_1 - 1}.$$

(Please turn to page 92)

STRUCTURAL ISSUES FOR HYPERPERFECT NUMBERS

DANIEL MINOLI
Bell Laboratories, Holmdel, NH 07733

ABSTRACT

An integer m is said to be n-hyperperfect if $m = 1 + n[\sigma(m) - m - 1]$. These numbers are a natural extension of the perfect numbers, and as such share remarkably similar properties. In this paper we investigate sufficient forms for hyperperfect numbers.

1. INTRODUCTION

Integers having "some type of perfection" have received considerable attention in the past few years. The most well-known cases are: perfect numbers ([1], [12], [13], [14], [15]); multiperfect numbers ([1]); quasiperfect numbers ([2]); almost perfect numbers ([3], [4], [5]); semiperfect numbers ([16], [17]);