A NEW ANGLE ON THE GEOMETRY OF THE FIBONACCI NUMBERS

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The "angle" we have in mind is a *gnomon*, a planar region that has the general shape of a carpenter's square. At the time of Pythagoras, a carpenter's square was in fact called a gnomon. The term came from Babylonia, where it originally referred to the vertically placed bar that cast the shadow on a sundial. The ancient Greeks also inherited a large body of algebra from the Babylonians, which they proceeded to recast into geometric terms. The gnomon became a recurrent figure in the Greek geometric algebra.

There are several reasons why Babylonian algebra was not adopted as it was, principally the discovery of irrationals: an irrational was acceptable to the Greeks as a length but not as a number. A secondary reason but, nevertheless, one of significance, was the Greek "delight in the tangible and visible" [2].

In this note we shall attempt to make the numbers $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, ... in the Fibonacci sequence "tangible and visible" by representing each F_m with a gnomon. These figures will enable us geometrically to derive or interpret many of the standard identities for the Fibonacci numbers. The ideas work equally well for the Lucas numbers and other generalized Fibonacci number sequences.

The gnomons we shall associate with the Fibonacci numbers are depicted in Figure 1. The angular shape that represents the mth Fibonacci number will be called the F_m*-gnomon*. In particular, "observe" the F_0 = 0-gnomon!

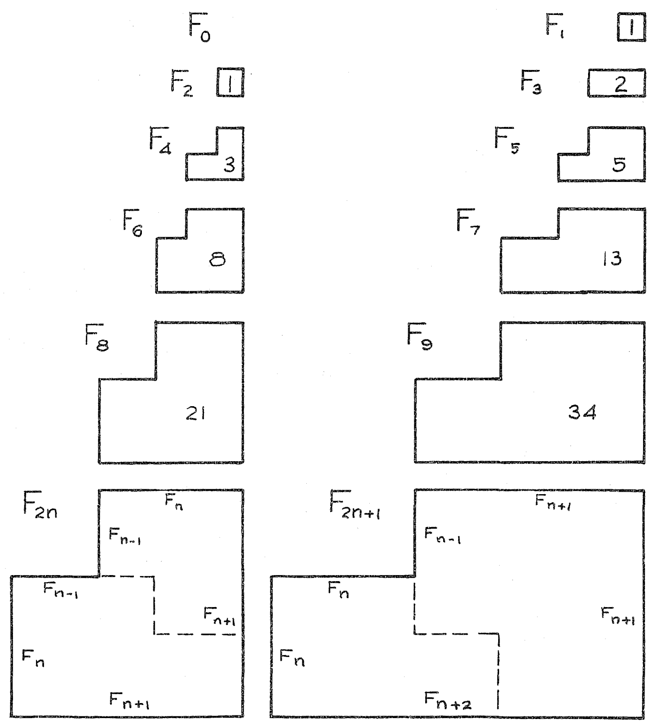


Fig. 1

The dashed lines in the lowermost gnomons indicate how the F_m- and F_{m+1}-gnomons can be combined to form the F_{m+2}-gnomon. This geometrically illustrates the basic recursion relation

$$F_{m+2} = F_{m+1} + F_m, \ m \geq 0. \tag{1}$$

The left-hand column of Figure 1 shows rather strikingly that the evenly indexed Fibonacci numbers are differences of squares of Fibonacci numbers. Indeed

$$F_{2n} = F_{n+1}^2 - F_{n-1}^2, \ n \geq 1. \tag{2}$$

Equally obvious from the right-hand column is the identity

$$F_{2n+1} = F_{n+1}^2 + F_n^2, \ n \geq 0. \tag{3}$$

Several other identities can be read off easily:

$$F_{2n+1} = F_{n-1}F_{n+1} + F_n F_{n+2}, \ n \geq 1; \tag{4}$$

$$F_{2n+1} = F_{n+2}F_{n+1} - F_n F_{n-1}, \ n \geq 1; \tag{5}$$

$$F_{2n} = F_{n-1}F_n + F_n F_{n+1}, \ n \geq 1. \tag{6}$$

Since $L_n = F_{n+1} + F_{n-1}$ is the nth Lucas number, it follows from (6) that

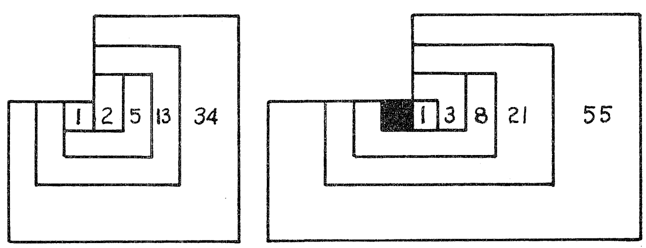
$$F_{2n} = L_n F_n, \ n \geq 1. \tag{7}$$

The gnomons in the left-hand column of Figure 1 can be superimposed in the manner shown in Figure 2(a). This shows how the F_{2n}-gnomon can be decomposed into "triple" gnomons of area $F_{2j} - F_{2j-2}$, $j = 1, \ldots, n$. From identity (1), we already know $F_{2j-1} = F_{2j} - F_{2j-2}$, and so

$$F_1 + F_3 + \cdots + F_{2n-1} = F_{2n}, \ n \geq 1. \tag{8}$$

In a similar manner (note the shaded unit square "hole") we see from Figure 2(b) that

$$F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1, \ n \geq 1. \tag{9}$$



(a) (b)

Fig. 2

We have noted that the F_{m+2}-gnomon can be dissected into an F_{m+1}- and F_m-gnomon. The larger of these can, in turn, be dissected into an F_m- and F_{m-1}-gnomon, and the larger of these can then be dissected into an F_{m-1}- and F_{m-2}-gnomon. Continuing this process dissects the original F_{m+2}-gnomon into a spiral that consists of the F_j-gnomons, $j = 1, \ldots, m$, together with an additional unit square (shown black), as illustrated in Figure 3. The separation of the gnomons into quadrants is rather unexpected.

From Figure 3, we conclude that

$$F_1 + F_2 + \cdots + F_m = F_{m+2} - 1, \ m \geq 0. \tag{10}$$

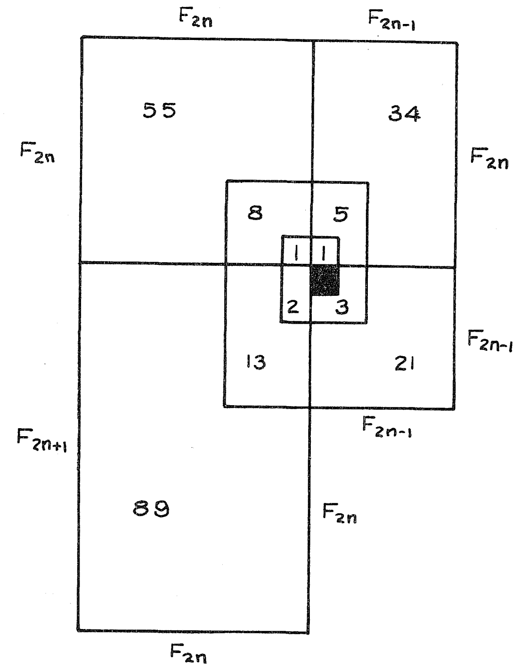


Fig. 3

Geometrically, we see that the first m Fibonacci gnomons can be combined with an additional unit square to form the F_{m+2}-gnomon. It is interesting to check this out successively for the special cases $m = 0, 1, 2, \ldots$.

The spiral pattern gives rise to additional identities. For example, by adding the areas of the gnomons in the first quadrant, we find

$$F_1 + F_5 + \cdots + F_{4n-3} = F_{2n-1}F_{2n}, \; n \geq 1. \tag{11}$$

The same procedure for the other three quadrants yields:

$$F_2 + F_6 + \cdots + F_{4n-2} = F_{2n}^2, \; n \geq 1; \tag{12}$$

$$F_3 + F_7 + \cdots + F_{4n-1} = F_{2n}F_{2n+1}, \; n \geq 1; \tag{13}$$

$$F_4 + F_8 + \cdots + F_{4n} = F_{2n+1}^2 - 1, \; n \geq 1. \tag{14}$$

The gnomons in the first quadrant are each a sum of two squares. (Some additional horizontal segments can be imagined in Figure 3.) We see that

$$F_1^2 + F_2^2 + \cdots + F_{2n-1}^2 = F_{2n-1}F_{2n}, \; n \geq 1. \tag{15}$$

Similarly, the third quadrant demonstrates

$$F_1^2 + F_2^2 + \cdots + F_{2n}^2 = F_{2n}F_{2n+1}, \; n \geq 1. \tag{16}$$

Of course, identities (15) and (16) are more commonly written simultaneously in the form

$$F_1^2 + F_2^2 + \cdots + F_m^2 = F_mF_{m+1}, \; m \geq 1. \tag{17}$$

Next, consider the F_{2n-1} by F_{2n+1} rectangle that the spiral covers in the right half-plane. Evidently, the area of this rectangle is one unit more than the area of the F_{2n} by F_{2n} square covered by the spiral in the third quadrant. Thus

$$F_{2n-1}F_{2n+1} = F_{2n}^2 + 1, \; n \geq 1. \tag{18}$$

An analogous consideration of the F_{2n} by F_{2n+2} rectangle covered by the spiral in the left half-plane shows

$$F_{2n}F_{2n+2} = F_{2n+1}^2 - 1, \; n \geq 1. \tag{19}$$

The black square at the center of the spiral plays an interesting role in the geometric derivation of these relations.

The geometric approach used above can be extended easily to deal with generalized Fibonacci sequences $T_1 = p$, $T_2 = q$, $T_3 = p+q$, $T_4 = p+2q$, ..., where p and q are positive integers. The T_m-gnomons can be taken as shown in Figure 4 (however, it should be mentioned that other gnomon shapes can be adopted, and will do just as well).

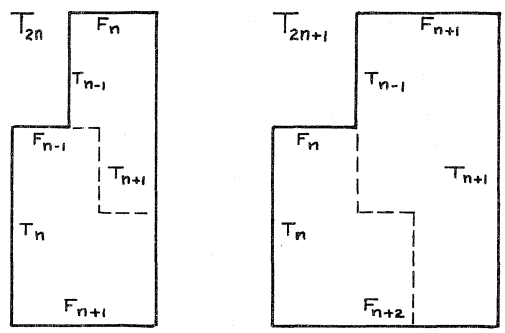


Fig. 4

From Figure 4, it is clear that

$$T_{m+2} = T_{m+1} + T_m, \; m \geq 1; \tag{20}$$

$$T_{2n} = T_{n-1}F_n + T_nF_{n+1}, \; n \geq 1; \tag{21}$$

$$T_{2n+1} = T_nF_n + T_{n+1}F_{n+1}, \; n \geq 1. \tag{22}$$

As before, a spiral pattern can be obtained readily. Figure 5 shows the spiral that corresponds to the Lucas sequence $L_1 = 1$, $L_2 = 3$, $L_3 = 4$, $L_4 = 7$, ..., where $p = 1$ and $q = 3$.

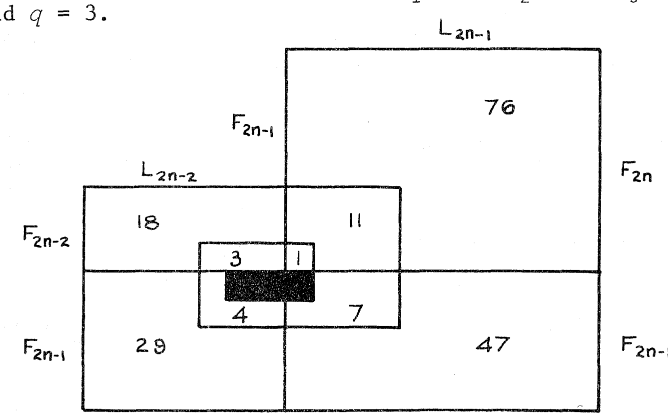


Fig. 5

It is clear from Figure 5 that

$$L_1 + L_2 + \cdots + L_m = L_{m+2} - 3,\ m \geq 1. \tag{23}$$

For the generalized sequence, one would find

$$T_1 + T_2 + \cdots + T_m = T_{m+2} - q,\ m \geq 1. \tag{24}$$

Beginning with a $q \times 1$ (black) rectangle, one can use identity (24) successively for $m = 1, 2, \ldots$ to generate T_m-gnomons. A variety of identities for generalized Fibonacci numbers can be observed and discovered by mimicking the procedures followed earlier.

It seems appropriate to conclude with a remark of Brother Alfred Brousseau: "It appears that there is a considerable wealth of enrichment and discovery material in the general area of Fibonacci numbers as related to geometry" [1]. Additional geometry of Fibonacci numbers can be found in Bro. Alfred's article.

REFERENCES

1. Brother Alfred Brousseau. "Fibonacci Numbers and Geometry." *The Fibonacci Quarterly* 10 (1972):303-318.
2. B. L. van der Waerden. *Science Awakening*, p. 125. New York: Oxford University Press, 1961.

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FIBONACCI AND LUCAS CUBES

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1. INTRODUCTION

The Fibonacci numbers are defined by the well-known recursion formulas

$$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$$

and the Lucas numbers by

$$L_0 = 2,\ L_1 = 1,\ L_n = L_{n-1} + L_{n-2}.$$

J. H. E. Cohn [2] determined the Fibonacci and Lucas numbers that are perfect squares. R. Finkelstein and H. London [3] gave a rather complicated determination of the cubes in the Fibonacci and Lucas sequences. Diophantine equations whose solutions must be Fibonacci and Lucas cubes occur in C. L. Siegel's proof [7] of H. M. Stark's result that there are exactly nine complex quadratic fields of class number one. This paper presents a simple determination of all Fibonacci numbers F_n of the form $2^a 3^b X^3$ and all Lucas numbers L_n of the form $2^a X^3$.

2. PRELIMINARY REDUCTIONS

From the recursion formulas defining the Fibonacci and Lucas numbers, it is easily verified by induction that the sequence of residues of F_n and L_n (mod p) are periodic, and in particular that

$$2 \mid F_n \text{ iff } 3 \mid n \tag{1}$$

$$2 \mid L_n \text{ iff } 3 \mid n \tag{2}$$

$$3 \mid F_n \text{ iff } 4 \mid n \tag{3}$$