THE DECIMAL EXPANSION OF 1/89 AND RELATED RESULTS

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One of the more bizarre and unexpected results concerning the Fibonacci sequence is the fact that

\[ \frac{1}{89} = 0.0112358 \]

\[
\begin{array}{c}
13 \\
21 \\
34 \\
55 \\
89 \\
144 \\
233 \\
\vdots
\end{array}
\]

which follows immediately from Binet's formula, as do the equations

\[ \frac{19}{89} = \sum_{i=1}^{m} \frac{L_i}{10^i} \]

\[ \frac{1}{109} = \sum_{i=1}^{m} \frac{F_i}{(-10)^i} \]

and

\[ \frac{21}{109} = \sum_{i=1}^{m} \frac{L_i}{(-10)^i} \]

where \( F_i \) denotes the \( i \)th Fibonacci number. The result follows immediately from Binet's formula, as do the equations

where \( L_i \) denotes the \( i \)th Lucas numbers. It is interesting that all these results can be obtained from the following unusual identity, which is easily proved by mathematical induction.

**Theorem 1:** Let \( a, b, c, d, \) and \( B \) be integers. Let \( \{\mu_n\} \) be the sequence defined by the recurrence \( \mu_0 = a, \mu_1 = d, \mu_{n+2} = a\mu_{n+1} + b\mu_n \) for all \( n \geq 2 \). Let \( m \) and \( N \) be integers defined by the equations

\[ B = m + Ba + b \quad \text{and} \quad N = cm + dB + bc. \]

Then

\[ B^n N = m \sum_{i=1}^{n+1} B^{n+1-i} \mu_i + B \mu_{i+1} + b \mu_i \]

for all \( n \geq 0 \). Also, \( N \equiv 0 \pmod{B} \).

**Proof:** The result is clearly true for \( n = 0 \), since it then reduces to the equation

\[ N = cm + dB + bc \]

of the hypotheses. Assume that

\[ B^{k+1} N = m \sum_{i=1}^{k+1} B^{k+2-i} \mu_i + B \mu_{i+1} + b \mu_i \]

Then
\[
\begin{align*}
\mu_n &= \left( \frac{c}{2} + \frac{2d - c}{\sqrt{a^2 + 4b}} \right) \left( \frac{a + \sqrt{a^2 + 4b}}{2} \right)^n + \left( \frac{c}{2} - \frac{2d - c}{\sqrt{a^2 + 4b}} \right) \left( \frac{a - \sqrt{a^2 + 4b}}{2} \right)^n.
\end{align*}
\] (6)

Thus it follows from (5) that
\[
\frac{N}{Bm} = \sum_{i=1}^{n+1} \frac{\mu_{i-1}}{B^i} + \frac{B_m \mu_{n+1} + b \mu_n}{mb^{n+1}} = \sum_{i=1}^{n} \frac{\mu_{i-1}}{B^i},
\] (7)

provided that the remainder term tends to 0 as \( n \) tends to infinity, and a sufficient condition for this is that
\[
\left| \frac{a + \sqrt{a^2 + 4b}}{2B} \right| < 1 \quad \text{and} \quad \left| \frac{a - \sqrt{a^2 + 4b}}{2B} \right| < 1.
\]

Thus we have proved the following theorem.

**Theorem 2:** If \( a, b, c, d, m, N, \) and \( B \) are integers, with \( m \) and \( N \) as defined above and if
\[
\left| \frac{a + \sqrt{a^2 + 4b}}{2B} \right| < 1 \quad \text{and} \quad \left| \frac{a - \sqrt{a^2 + 4b}}{2B} \right| < 1,
\]

then
\[
\frac{N}{Bm} = \sum_{i=1}^{n} \frac{\mu_{i-1}}{B^i}.
\] (8)

Of course, equations (1)-(4) all follow from (8) by particular choices of \( a, b, c, \) and \( d \). To obtain (2), for example, we set \( a = 2, \) \( b = 1, \) and \( B = 10 \). It then follows that
\[
m = B^2 - Ba - b = 100 - 10 - 1 = 89
\]
\[
N = cm + dB + bc = 178 + 10 + 2 = 190
\]
and
\[
\frac{19}{89} = \frac{190}{10 \cdot 89} = \frac{N}{Bm} = \sum_{i=1}^{n} \frac{L_{i-1}}{10^i} \quad \text{as claimed.}
\]

To obtain (3), we set \( a = 0, b = 1, \) and \( B = -10 \). Then
\[
m = B^2 - Ba - b = 100 + 10 - 1 = 109,
\]
\[
N = cm + dB + bc = -10,
\]
and
\[
\frac{N}{Bm} = \frac{-10}{-10 \cdot 109} = \frac{1}{109} = \sum_{i=1}^{n} \frac{L_{i-1}}{(10)^i} \quad \text{as indicated.}
\]
Finally, we note that interesting results can be obtained by setting $B$ equal to a power of 10. For example, if $B = 10^h$ for some integer $h$, $a = 0$, and $a = b = d = 1$,

$$m = 10^{2h} - 10^h - 1, \quad H = 10^h,$$

and (8) reduces to

$$\frac{1}{10^{2h} - 10^h - 1} = \sum_{i=1}^{m} \frac{F_{i-1}}{10^{hi}}.$$  \hspace{1cm} (9)

For successive values of $h$ this gives

$$\frac{1}{99} = \sum_{i=1}^{m} \frac{F_{i-1}}{10^{2hi}},$$  \hspace{1cm} (10)

as we already know,

$$\frac{1}{9899} = \sum_{i=1}^{m} \frac{F_{i-1}}{10^{2hi}} = .000101020305081321\ldots,$$  \hspace{1cm} (11)

$$\frac{1}{998999} = \sum_{i=1}^{m} \frac{F_{i-1}}{10^{3hi}} = .000001001002003005008013\ldots,$$  \hspace{1cm} (12)

and so on. In case $B = (-10)^h$ for successive values of $h$, $a = 0$, and $a = b = d = 1$, we obtain

$$\frac{1}{109} = \sum_{i=1}^{m} \frac{F_{i-1}}{(-10)^i},$$  \hspace{1cm} (13)

$$\frac{1}{10099} = \sum_{i=1}^{m} \frac{F_{i-1}}{(-100)^i},$$  \hspace{1cm} (14)

$$\frac{1}{1000999} = \sum_{i=1}^{m} \frac{F_{i-1}}{(-1000)^i},$$  \hspace{1cm} (15)

and so on. Other fractions corresponding to (2) and (3) above are

$$\frac{19}{89}, \frac{199}{9899}, \frac{1999}{998999}, \ldots$$

and

$$\frac{21}{109}, \frac{201}{10099}, \frac{2001}{1000999}, \ldots.$$