ADVANCED PROBLEMS AND SOLUTIONS

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Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, Mathematics Department, Lock Haven State College, Lock Haven, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, the solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-327 *Proposed by James F. Peters, St. John's University, Collegeville, MN*

The sequence

1,3,4,6,8,9,11,12,14,16,17,19,21,22,24,25,27,29,30,32,34,35, ...

was introduced by D. E. Thoro [Advanced Problem H-12, *The Fibonacci Quarterly* 1 (1963):54]. Dubbed "A curious sequence," the following is a slightly modified version of the defining relation for this sequence suggested by the Editor (*The Fibonacci Quarterly* 1 (1963):50):

If

$$T_0 = 1,\ T_1 = 3,\ T_2 = 4,\ T_3 = 6,\ T_4 = 8,\ T_5 = 9,\ T_6 = 11,\ T_7 = 12,$$

then

$$T_{8m+k} = 13m + T_k,\ \text{where } k \geq 0,\ m = 1, 2, 3, \ldots .$$

Assume

$$F_0 = 1,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1}$$

and verify the following identities:

(1) $T_{F_n - 2} = F_{n+1} - 2$, where $n \geq 6$.

For example,

$$T_{F_6 - 2} = T_6 = 11 = F_7 - 2$$
$$T_{F_7 - 2} = T_{11} = 19 = F_8 - 2$$
etc.

(2) $T_{F_n - 2} - T_{F_{n-2} - 2} = F_n$, where $n \geq 6$.

(3) $T_{F_n - 2} = F_{n+1} - 2 + L_{n-12}$, where $n \geq 15$.

H-328 *Proposed by Verner E. Hoggatt, Jr.*

Let θ be a positive irrational number such that $1/\theta + 1/\theta^{j+1} = 1$ ($j \geq 1$ an integer) Further, let

$$A_n = [n\theta],\ B_n = [n\theta^{j+1}],\ \text{and } C_n = [n\theta^j].$$

Prove: (a) $A_{C_n} + 1 = B_n$

(b) $A_{C_n+1} - A_{C_n} = 2$

$A_{m+1} - A_m = 1$ ($m \neq C_k$ for any $k > 0$)

(c) $B_n - n$ is the number of A_j's less than B_n.

H-329 *Proposed by Leonard Carlitz, Duke University, Durham, NC*

Show that, for s and t nonnegative integers,

$$(1)\quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k}{s}\binom{k}{t} = \sum_k \frac{x^{s+t-k}}{k!(s-k)!(t-k)!}.$$

More generally, show that

$$(2)\quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k+\alpha}{s}\binom{k}{t} = \sum_k \frac{x^{s+t-k}}{(s-k)!t!}\binom{\alpha+t}{k}$$

and

$$(3)\quad e^{-x}\sum_k \frac{x^k}{k!}\binom{k}{s}\binom{k+\beta}{t} = \sum_k \frac{x^{s+t-k}}{s!(t-k)!}\binom{\beta+s}{k}.$$

SOLUTIONS

Determined

H-302 *Proposed by George Berzsenyi, Lamar University, Beaumont, TX (Vol. 17, No. 3, October 1979)*

Let c be a constant and define the sequence $\langle a_n \rangle$ by $a_0 = 1$, $a_1 = 2$, and $a_n = 2a_{n-1} + ca_{n-2}$ for $n \geq 2$. Determine the sequence $\langle b_n \rangle$ for which

$$a_n = \sum_{k=0}^{n}\binom{n}{k}b_k.$$

Solution by the proposer.

The equation $a_n = \sum_{k=0}^{n}\binom{n}{k}b_k$ determines the sequence $\langle b_n \rangle$ uniquely as it is easily seen by letting $n = 0, 1, 2, \ldots$ in succession and solving the resulting equalities recursively for $b_0, b_1, b_2, \ldots$. The first few values are thus found to be

$$b_0 = 1,\ b_1 = 1,\ b_2 = c + 1,\ b_3 = c + 1,\ b_4 = (c + 1)^2,\ \ldots .$$

We will prove that the sequence $\langle b_n \rangle$ defined by $b_{2n} = b_{2n+1} = (c + 1)^n$ satisfies the given equation and envoke its unicity to solve the problem.

The generating functions $A(x)$ and $B(x)$ for the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$, respectively, are easily shown to be

$$A(x) = \frac{1}{1 - 2x - cx^2} \quad\text{and}\quad B(x) = \frac{1 + x}{1 - x^2 - cx^2}.$$

Therefore, utilizing Hoggatt's approach [*The Fibonacci Quarterly* 9 (1971):122], one finds

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}b_k x^n = \frac{1}{1-x}\sum_{n=0}^{\infty}b_n\left(\frac{x}{1-x}\right)^n = \frac{1}{1-x}\,\frac{1 + \dfrac{x}{1-x}}{1 - \left(\dfrac{x}{1-x}\right)^2 - c\left(\dfrac{x}{1-x}\right)^2}$$

$$= \frac{1}{1 - 2x - cx^2} = \sum_{n=0}^{\infty}a_n x^n,$$

implying the desired relationship between the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$.

Also solved by P. Bruckman, P. Byrd, D. Russell, and A. Shannon.

Zeta

H-303 *Proposed by Paul Bruckman, Concord, CA*
(Vol. 17, No. 3, October 1979)

If $0 < s < 1$, and n is any positive integer, let

(1) $$H_n(s) = \sum_{k=1}^{n} k^{-s},$$

and

(2) $$\theta_n(s) = \frac{n^{1-s}}{1-s} - H_n(s).$$

Prove that $\lim\limits_{n\to\infty} \theta_n(s)$ exists, and find this limit.

Solution by the proposer.

The following is Formula 23.2.9 in *Handbook of Mathematical Functions*, ed. by M. Abramowitz and I. A. Stegun. Ninth Printing. (Washington, D.C.: National Bureau of Standards, Nov. 1970 [with corrections]), p. 807:

(3) $$\zeta(s) = \sum_{k=1}^{n} k^{-s} + (s-1)^{-1}n^{1-s} - s\int_n^\infty \frac{x-[x]}{x^{s+1}}dx,\ n = 1, 2, \ldots;\ s \neq 1,\ \operatorname{Re}(s) > 0,$$

where ζ is the Riemann zeta function. If we let

(4) $$I_n(s) = \int_n^\infty \frac{x-[x]}{x^{s+1}}dx,$$

we see that formula (3) reduces to

(5) $$-\zeta(s) = \theta_n(s) + sI_n(s).$$

Note from (4) that $I_n(s) > 0$. Moreover,

$$I_n(s) < \int_n^\infty \frac{dx}{x^{s+1}} = \frac{1}{sn^s}.$$

Hence, $\lim\limits_{n\to\infty} sI_n(s) = \lim\limits_{n\to\infty} n^{-s} = 0$. We thus see from (5) that

(6) $$\lim_{n\to\infty} \theta_n(s) = -\zeta(s).$$

Since $\zeta(s)$ is defined for $0 < s < 1$, this is the solution to the problem.

Like Fibonacci-like Sum

H-305 *Proposed by Martin Schechter, Swarthmore College, Swarthmore, PA*
(Vol. 17, No. 3, October 1979)

For fixed positive integers m and n, define a Fibonacci-like sequence as follows:

$$S_1 = 1,\ S_2 = m,\ S_k = \begin{cases} mS_{k-1} + S_{k-2} & \text{if } k \text{ is even,} \\ nS_{k-1} + S_{k-2} & \text{if } k \text{ is odd.} \end{cases}$$

(Note that for $m = n = 1$, one obtains the Fibonacci numbers.)

(a) Show the Fibonacci-like property holds that if j divides k then S_j divides S_k and in fact that $(S_q, S_r) = S_{(q,r)}$ where (,) = g.c.d.

(b) Show that the sequences obtained

$$\text{when } [m = 1,\ n = 4] \text{ and when } [m = 1,\ n = 8],$$

respectively, have only the element 1 in common.

Partial solution by the proposer.

(a) It is convenient first to define a sequence of polynomials $\{Q_k\}_1^\infty$, where Q_k is a polynomial of k commuting variables, as follows:

$$Q_0 = 1,\ Q_1(a_1) = a_1,$$

and

$$Q_k(a_1, \ldots, a_k) = a_k Q_{k-1}(a_1, \ldots, a_{k-1}) + Q_{k-2}(a_1, \ldots, a_{k-2}).$$

It is easy to show by induction that for $j = 1, \ldots, k - 1$, Q_k has the expansion:

$$Q_k(a_1, \ldots, a_k) = Q_j(a_1, \ldots, a_j)Q_{k-j}(a_{j+1}, \ldots, a_k) - Q_{j-1}(a_1, \ldots, a_{j-1})Q_{k-j-1}(a_{j+2}, \ldots, a_k).$$

Note that $S_k = Q_{k-1}\underbrace{(m, n, m, n, \ldots)}_{k-1}$

Associated to S_k is the sequence $\overline{S}_k$, which is obtained by interchanging the roles of m and n. The sequences S_k and $\overline{S}_k$ are easily shown to satisfy the relations:

$$S_k = \overline{S}_k \quad \text{if } k \text{ is odd,}$$

$$nS_k = m\overline{S}_k \quad \text{if } k \text{ is even.}$$

Note that if j is odd, $S_j = (mn + 1)S_{j-2} + nS_{j-3}$.

It follows from this equation, by induction, that if j is odd, then $(S_j, n) = 1$. It is also clear that for any j, $(S_j, S_{j+1}) = 1$.

Using the above polynomials, we may readily establish:

$$S_k = \begin{cases} S_{j+1}S_{k-j} + S_j\overline{S}_{k-j-1} & \text{if } j \text{ is even,} \\ S_{j+1}\overline{S}_{k-j} + S_jS_{k-j-1} & \text{if } j \text{ is odd.} \end{cases}$$

An easy induction argument now shows that $j|k$ implies $S_j|S_k$.

Finally, an indirect argument using induction shows that

$$(S_q, S_r) = S_{(q,r)}.$$

<u>*Late Acknowledgment*</u>: H-281 *solved by J. Shallit*, H-283 *solved by J. La Grange*.

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