FIBONACCI NUMBERS AND STOPPING TIMES

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For each integer \( k \geq 2 \), let \( \{a_{n,k}\} \) and \( \{b_{n,k}\} \) be two sequences of integers defined by \( a_{n,k} = 0 \) for all \( n = 1, \ldots, k - 1 \), \( a_{k,k} = 1 \), and

\[
a_{n,k} = \sum_{j=1}^{k} a_{n-j,k}
\]

for all \( n \geq k \); \( b_{1,k} = 0 \), and

\[
b_{n,k} = a_{n,k} + \sum_{j=1}^{n-1} a_{j,k} b_{n-j,k}
\]

for all \( n \geq 2 \).

Let \( \{Y_n\} \) be the fair coin-tossing sequence, i.e.,

\[
P(Y_j = 0) = \frac{1}{2} = P(Y_j = 1)
\]

for all \( j = 1, 2, \ldots \), and \( Y_1, Y_2, \ldots \) are independent. With respect to the sequence \( \{Y_n\} \), for each integer \( k \geq 1 \), let \( \{R_{n,k}\} \) and \( \{N_{n,k}\} \) be two sequences of stopping times defined by

\[
R_{1,k}(Y_1, Y_2, \ldots) = \inf \{m|Y_m = \cdots = Y_{m-k+1} = 0\},
\]

= \infty if no such \( m \) exists, and for all \( n \geq 2, \)

\[
R_{n,k}(Y_1, Y_2, \ldots) = \inf \{m|m \geq R_{n-1,k} + k \text{ and } Y_m = \cdots = Y_{m-k+1} = 0\},
\]

= \infty if no such \( m \) exists; \( N_{1,k} = R_{1,k} \) and \( N_{n,k} = R_{n,k} - R_{n-1,k} \) for all \( n \geq 2 \).

In this note, we shall prove the following interesting theorems.

**Theorem 1:** For each integer \( k \geq 2 \),

\[a_{n,k} = 2^n P(N_{1,k} = n) \text{ and } b_{n,k} = 2^n P(R_{m,k} = n \text{ for some integer } m \geq 1).\]

**Theorem 2:** For each integer \( k \geq 2 \),

\[b_{n,k} = 2b_{n-1,k} + 1 \text{ or } 2b_{n-1,k} - 1 \text{ or } 2b_{n-1,k}\]

according as \( n = mk \) or \( mk + 1 \) or \( mk + j \) for some integers \( m \geq 1 \) and \( j = 2, 3, \ldots, k - 1 \).

**Theorem 3:** For each integer \( k \geq 2 \), let

\[
\mu_k = \sum_{n=1}^{m} n 2^{-n} a_{n,k} = E(N_{1,k}),
\]

then

\[b_{nk,k} = (2^{nk} + 2^{k} - 2) / \mu_k \text{ and } b_{mk+j,k} = 2^{j-1}(2^{nk+1} - 2) / \mu_k\]

for all \( n \geq 1 \) and \( j = 1, 2, \ldots, k - 1 \).

We start with the following elementary lemmas.

**Lemma 1:** For each integer \( k \geq 1 \), let

\[
\phi_k(t) = E(t N_{1,k}) \text{ if } E(\mid t \mid N_{1,k}) < \infty;
\]

then

\[
\phi_k(t) = \left(\frac{t}{2}\right)^k \left\{ 1 - \sum_{j=1}^{k} \left(\frac{t}{2}\right)^j \right\} \text{ for all } -1 \leq t \leq 1.
\]
Proof: For $k = 1$, it is well known that
\[
\Phi_1(t) = \left(\frac{t}{2}\right) \left\{ 1 - \left(\frac{t}{2}\right) \right\}
\]
for all $t$ in $[-1, 1]$. For $k \geq 2$, it is easy to see that $N_{1,k} = N_{1,k-1} + Z$, where $Z$ is a random variable such that
\[
P(Z = 1) = P(Z = 1 + N_{1,k}) = \frac{1}{2}
\]
and $Z$ is independent of $N_{1,k-1}$. Hence
\[
\Phi_i(t) = \frac{t}{2} \left\{ \Phi_{i-1}(t) + \Phi_i(t) \right\}
\]
for all $-2 < t < 2$. Therefore, for each integer $k \geq 1$,
\[
\Phi_k(t) = \left(\frac{t}{2}\right) \left\{ 1 - \sum_{j=1}^{k} \left(\frac{t}{2}\right) \right\}
\]
for all $-1 \leq t \leq 1$.

Lemma 2: For each integer $k \geq 2$, let
\[
G_k(t) = \sum_{n=1}^{\infty} t^n a_{n,k}
\]
for all $t$ such that
\[
\sum_{n=1}^{\infty} |t|^n a_{n,k} < \infty;
\]
then
\[
G_k(t) = \Phi_k(2t) \text{ for all } -\frac{1}{2} < t < \frac{1}{2} \text{ and } k \geq 2.
\]
Proof: Since $a_{n,k} = 0$ for all $n = 1, 2, \ldots, k-1, a_{k,k} = 1$, and
\[
a_{n,k} = \sum_{j=1}^{k} a_{n-j,k}
\]
for all $n > k$,
\[
G_k(t) = \sum_{n=k}^{\infty} t^n a_{n,k} = t^k + \sum_{i=1}^{k} t^i \sum_{n=k}^{\infty} t^n a_{n,k} = t^k + \sum_{i=1}^{k} t^i G_i(t)
\]
Therefore,
\[
G_k(t) = t^k \left\{ 1 - \sum_{j=1}^{k} t^j \right\}
\]
for all $k \geq 2$ and all $t$ such that
\[
\sum_{n=1}^{\infty} |t|^n a_{n,k} < \infty.
\]
Since $a_{n,k} \leq 2^n$ for all $n \geq 1$ and all $k \geq 2$, $G_k(t)$ exists for all $-\frac{1}{2} < t < \frac{1}{2}$. By Lemma 1, we have
\[
G_k(t) = \Phi_k(2t) \text{ for all } t \text{ in the interval } \left( -\frac{1}{2}, \frac{1}{2} \right) \text{ and all } k \geq 2.
\]
For each integer $k \geq 1$, let $u_{0,k} = 1$, and for all $n \geq 1$, let
\[
u_{n,k} = P(R_m, k = n \text{ for some integer } m \geq 1)
\]
and $f_{n,k} = P(N_1, k = n)$. Since $\{Y_n\}$ is a sequence of i.i.d. random variables, and $u_{0,k} = 1$, it is easy to see that
\[
u_{n,k} = \sum_{j=1}^{n} f_{j,k} u_{n-j,k} \text{ for all } n \geq 1 \text{ and all } k \geq 1.
\]
Hence we have the following theorem.
**Theorem 1'**: For each integer \( k \geq 2 \), \( 2^n u_{n,k} = 2^n \mathbb{P}(R_{n,k} = n) = \alpha_{n,k} \) for some integer \( m \geq 1 \).

Let \( A = \{(\omega_1, \omega_2, \ldots, \omega_n) | \omega_k = 0 \text{ or } 1 \text{ for all } i = 1, 2, \ldots, n \text{ and } \omega_j = 1 \neq \omega_{j+1} = \cdots = \omega_n = 0 \text{ for some } j = n - jk \text{ and some integer } j \geq 1 \} \).

Let \( B = \{(v_1, v_2, \ldots, v_{n-1}) | v_k = 0 \text{ or } 1 \text{ for all } i = 1, 2, \ldots, n - 1 \text{ and } v_{j-1} = 1 \neq v_j = \cdots = v_{n-1} = 0 \text{ for some } j = n - jk \text{ for some integer } j \geq 1 \} \).

**Lemma 3**: For each integer \( k \geq 2 \),
\[ 2^n u_{n,k} = 2^n u_{n-1,k} + 1 \text{ or } 2^n u_{n-1,k} - 1 \text{ or } 2^n u_{n-1,k} \]
according as \( n = mk \) or \( mk + 1 \) or \( mk + j \) for some integers \( m \geq 1 \) and \( j = 2, 3, \ldots, k - 1 \).

**Proof**: By the definition of \( \{u_{n,k}\} \), for each integer \( k \geq 2 \),
\[ 2^n u_{n,k} = \text{the number of elements in } A \]
and
\[ 2^{n-1} u_{n-1,k} = \text{the number of elements in } B. \]

(i) If \( n = mk \) for some integer \( m \geq 1 \), then \((0, v_1, v_2, \ldots, v_{n-1})\) and \((1, v_1, v_2, \ldots, v_{n-1})\) are in \( A \) if \((v_1, v_2, \ldots, v_{n-1})\) is in \( B \), and \((0, 0, \ldots, 0)\), \((n - 1)\)-tuple, is also in \( A \) even \((0, 0, \ldots, 0)\), \((n - 1)\)-tuple, is not in \( B \). Hence the number of elements in \( A \geq 2 \cdot \) the number of elements in \( B + 1 \). Since each element \((\omega_1, \omega_2, \ldots, \omega_n)\) in \( A \) such that \( \omega_j \neq \omega_{j+1} \) for some \( 1 \leq j \leq n - 1 \) is a form of \((0, v_1, v_2, \ldots, v_{n-1})\) or a form of \((1, v_1, v_2, \ldots, v_{n-1})\) for some element \((v_1, v_2, \ldots, v_{n-1})\) in \( B \). Hence the number of elements in \( A \leq 2 \cdot \) the number of elements in \( B + 1 \). Therefore, the number of elements in \( A = 2 \cdot \) the number of elements in \( B + 1 \).

(ii) If \( n = mk + 1 \) for some integer \( m \geq 1 \), then \((0, v_1, v_2, \ldots, v_{n-1})\) and \((1, v_1, v_2, \ldots, v_{n-1})\) are in \( A \) if \((v_1, v_2, \ldots, v_{n-1})\) is in \( B \) and \( v_{j} \neq v_{j+1} \) for some \( 1 \leq j \leq n - 2 \) and \((1, 0, 0, \ldots, 0)\), \((n - 1)\)-tuple, is also in \( A \) \((0, 0, \ldots, 0)\), \((n - 1)\)-tuple, is not in \( B \). Hence the number of elements in \( A \geq 2 \cdot \) the number of elements in \( B - 1 \). Since each element \((\omega_1, \omega_2, \ldots, \omega_n)\) in \( A \) such that \( \omega_j \neq \omega_{j+1} \) for some \( 2 \leq j \leq n - 1 \) is a form of \((0, v_1, v_2, \ldots, v_{n-1})\) or a form of \((1, v_1, v_2, \ldots, v_{n-1})\) for some element \((v_1, v_2, \ldots, v_{n-1})\) in \( B \). Hence the number of elements in \( A \leq 2 \cdot \) the number of elements in \( B - 1 \). Therefore, the number of elements in \( A = 2 \cdot \) the number of elements in \( B - 1 \).

(iii) If \( n = mk + j \) for some integers \( m \geq 1 \) and \( 2 \leq j \leq k - 1 \), then \((0, v_1, v_2, \ldots, v_{n-1})\) and \((1, v_1, v_2, \ldots, v_{n-1})\) are in \( A \) if and only if \((v_1, v_2, \ldots, v_{n-1})\) is in \( B \). Therefore, the number of elements in \( A = 2 \cdot \) the number of elements in \( B \).

By (i), (ii), and (iii), the proof of Lemma 3 is now complete.

**Theorem 2'**: For each integer \( k \geq 2 \),
\[ b_{n,k} = 2b_{n-1,k} + 1 \text{ or } 2b_{n-1,k} - 1 \text{ or } 2b_{n-1,k} \]
according as \( n = mk \) or \( mk + 1 \) or \( mk + j \) for some integers \( m \geq 1 \) and \( j = 2, 3, \ldots, k - 1 \).

**Proof**: By Theorem 1' and Lemma 3.

For each integer \( k \geq 1 \), let
\[ \nu_k = E[N_{1,k}] = \sum_{n=1}^{\infty} \mathbb{P}(N_{1,k} = n) = \sum_{n=1}^{\infty} n f_{n,k}. \]

By Theorem 1',
\[ \mu_k = \sum_{n=1}^\infty n^2 \alpha_{n,k} \] for each integer \( k \geq 2 \).

Since
\[
\left( \frac{1}{2} \right)^k = \sum_{j=0}^{k-1} \mu_{n-j,k} \left( \frac{1}{2} \right)^j \quad \text{for all } n \geq k \text{ and } k \geq 1,
\]
\[
\left( \frac{1}{2} \right)^k = \lim_{n \to \infty} \sum_{j=0}^{k-1} \mu_{n-j,k} \left( \frac{1}{2} \right)^j.
\]

By the Renewal Theorem (see [1, p. 330]), we have
\[
\left( \frac{1}{2} \right)^k = (E(N_k,k))^{-1} \sum_{j=0}^{k-1} \left( \frac{1}{2} \right)^j \quad \text{for all } k \geq 1.
\]

Hence
\[
\mu_k = E(N_k,k) = \sum_{n=1}^\infty n^2 \alpha_{n,k} = \sum_{n=1}^\infty n^2 \alpha_{n,k} = \sum_{j=1}^{k} 2^j = 2^{k+1} - 2.
\]

**Theorem 3':** For each integer \( k \geq 2 \), let
\[
\mu_k = \sum_{n=1}^\infty n^2 \alpha_{n,k} = 2^{k+1} - 2;
\]
then
\[
b_{mk,k} = (2^{mk} + 2^k - 2)/\mu_k \quad \text{and} \quad b_{mk+j,k} = 2^j (2^{mk+1} - 2)/\mu_k
\]
for all integers \( m \geq 1 \) and \( j = 1, 2, \ldots, k - 1 \).

**Proof:** By the definition of \( \{b_{n,k}\} \), \( b_{k,k} = 1 \). Hence, by Theorem 2', Theorem 3' holds when \( m = 1 \). Suppose that Theorem 3' holds for \( m = 1, 2, \ldots, M - 1 \) and \( j = 1, 2, \ldots, k - 1 \), where \( M \) is an integer \( \geq 2 \). Now, let \( m = M \), then, by Theorem 2',
\[
b_{MK,k} = 2b_{MK-1,k} + 1 = 2^{k+1} (2^{M-1} k + 1 - 2)/\mu_k + 1 = (2^{MK} - 2^k + 2^{k+1} - 2)/\mu_k
\]
\[= (2^{MK} + 2^k - 2)/\mu_k,\]

since \( \mu_k = 2^{k+1} - 2 \).

\[
b_{MK+1,k} = 2b_{MK,k} - 1 = (2^{MK+1} + 2^{k+1} - 4)/\mu_k - 1 = (2^{MK+1} + 2^k - 2)/\mu_k.
\]

\[
b_{MK+j,k} = 2^{j-1} (2^{MK+1} - 2)/\mu_k, \quad \text{for all } j = 2, 3, \ldots, k - 1.
\]

Hence Theorem 3' holds for \( m = M \) and \( j = 1, 2, \ldots, k - 1 \). Therefore, Theorem 3' holds for all \( m \geq 1 \) and \( j = 1, 2, \ldots, k - 1 \).

**Corollary to Theorem 3':** For each integer \( k \geq 2 \),
\[
\mu_{mk,k} = \mu_k^{-1} \left( 1 + 2^{-mk+k} - 2^{-mk+1} \right) = (2^{k+1} - 2)^{-1} \left( 1 + 2^{-mk+k} - 2^{-mk+1} \right)
\]
and
\[
\mu_{mk+j,k} = \mu_k^{-1} \left( 1 - 2^{-mk} \right) = (2^{k+1} - 2)^{-1} \left( 1 - 2^{-mk} \right)
\]
for all integers \( m \geq 1 \) and \( j = 1, 2, \ldots, k - 1 \).

**Reference**


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