ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E., Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers \( F_n \) and Lucas numbers \( L_n \) satisfy

\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1
\]

and

\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.
\]

Also, \( a \) and \( b \) designate the roots \((1 + \sqrt{5})/2\) and \((1 - \sqrt{5})/2\), respectively, of \( x^2 - x - 1 = 0 \).

PROBLEMS PROPOSED IN THIS ISSUE

B-448 Proposed by Herta T. Freitag, Roanoke, VA

Prove that, for all positive integers \( t \),

\[
\sum_{i=1}^{2t} F_{5i+1} L_{5i} \equiv 0 \pmod{5}.
\]

B-449 Proposed by Herta T. Freitag, Roanoke, VA

Prove that, for all positive integers \( t \),

\[
\sum_{i=1}^{2t} (-1)^{i+1} F_{5i+1} L_{5i} \equiv 0 \pmod{7}.
\]

B-450 Proposed by Lawrence Somer, Washington, D.C.

Let the sequence \( \{H_n\}_{n=0}^\infty \) be defined by \( H_n = F_{2n} + F_{2n+2} \).

(a) Show that 5 is a quadratic residue modulo \( H_n \) for \( n \geq 0 \).

(b) Does \( H_n \) satisfy a recursion relation of the form \( H_{n+2} = cH_{n+1} + dH_n \), with \( c \) and \( d \) constants? If so, what is the relation?

B-451 Proposed by Keats A. Pullen, Jr., Kingsville, MD

Let \( k, m \), and \( p \) be positive integers with \( p \) an odd prime. Show that in base \( 2p \) the units digit of \( m^{k(p-1)+1} \) is the same as the units digit of \( m \).

B-452 Proposed by P. L. Mana, Albuquerque, NM

Let \( a_0 + a_1 x + a_2 x^2 + \cdots \) be the Maclaurin expansion for \( (1 - ax)(1 - bx) \)^{-1}, where \( a \neq b \). Find the rational function whose Maclaurin expansion is

\[
a_0^2 + a_1^2 x + a_2^2 x^2 + \cdots
\]

and use this to obtain the generating functions for \( F_n^2 \) and \( L_n^2 \).

B-453 Proposed by Paul S. Bruckman, Concord, CA

Solve in integers \( r, s, t \) with \( 0 \leq r < s < t \) the Fibonacci Diophantine equation
and the analogous Lulucas equation in which each \( F \) is replaced by an \( L \).

**SOLUTIONS**

**Counting Hands**

B-424 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, NM

Of the \( \binom{52}{5} \) possible 5-card poker hands, how many form any:

(i) full house?

(ii) flush?

(iii) straight?

Solution by Paul S. Bruckman, Concord, CA

(i) The two denominations represented in a full house may be chosen in \( 2 \left( \binom{13}{2} \right) \) ways, the coefficient "2" reflecting the fact that the three-of-a-kind may appear in either of two ways. The individual cards for these denominations can be chosen in \( \binom{4}{3} \binom{4}{2} \) ways. Thus, the total number of possible full houses is \( 2 \left( \binom{13}{2} \binom{4}{3} \binom{4}{2} \right) = 3,744 \).

(ii) The suit represented in a flush may be chosen in 4 ways, and the 5 cards of the flush in that suit may be chosen in \( \binom{13}{5} \) ways. Hence, the total number of possible flushes (including "straight flushes") is \( 4 \binom{13}{5} = 5,148 \).

(iii) With the ace being either high or low, there are 10 different ways to choose the denominations appearing in a straight. With each of these ways, there are \( 4^5 \) choices for the individual cards. Thus, the total number of possible straights (including "straight flushes") is \( 10 \cdot 4^5 = 10,240 \).

**NOTE:** Since the total number of possible straight flushes is 10 \( \cdot 4 = 40 \), the answers to (ii) and (iii) above excluding straight flushes would be reduced by 40, and so would equal 5,108 and 10,200, respectively.

Also solved by John W. Milsom, Bob Priall, Charles B. Shields, Lawrence Somer, Gregory Wulczyn, and the proposer.

**Average in a Fixed Rank**

B-425 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, NM

Let \( k \) and \( n \) be positive integers with \( k < n \), and let \( S \) consist of all \( k \)-tuples \( X = (x_1, x_2, \ldots, x_k) \) with each \( x_j \) an integer and

\[
1 \leq x_1 < x_2 < \cdots < x_k \leq n.
\]

For \( j = 1, 2, \ldots, k \), find the average value \( \bar{x}_j \) of \( x_j \) over all \( X \) in \( S \).

Solution by Graham Lord, Université Laval, Québec, Canada

The number of \( k \)-tuples \( X \) in which \( x_j = m \) is \( \binom{m-1}{j-1} \binom{n-m}{k-j} \). [Choose the \( j-1 \) smaller integers from among the first \( m-1 \) natural numbers and the \( k-j \) larger ones from among \( m+1, \ldots, n \).] Evidently the total number of \( k \)-tuples,

\[
\sum_{m \geq j} \binom{m-1}{j-1} \binom{n-m}{k-j}.
\]
is simply the number of \(k\)-subsets from \(\{1, 2, \ldots, n\}\), that is \(\binom{n}{k}\). Hence,

\[
\frac{n^2}{k} = \left\{ \frac{n^2}{\binom{n}{k}} \right\} = \frac{j}{j + 1},
\]

Also solved by Paul S. Bruckman and the proposer.

**Fibonacci Pythagorean Triples**

B-426 Proposed by Herta T. Frietag, Roanoke, VA

Is \((F_n F_{n+1})^2 + (2F_{n+1} F_{n+2})^2\) a perfect square for all positive integers \(n\), i.e., are there integers \(\sigma_n\) such that \((F_n F_{n+3}, 2F_{n+1} F_{n+2}, \sigma_n)\) is always a Pythagorean triple?

Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI

The answer to the question posed above is "yes" and \(\sigma_n = F_{2n+3}\). To establish this result, we observe that

\[E_n F_{n+3} = (F_{n+2} - F_{n+1})(F_{n+2} + F_{n+1}) = E_{n+2}^2 - E_{n+1}^2\]

so

\[2E_{n+1}E_{n+2} + (2F_{n+1} F_{n+2})^2 = (E_{n+2}^2 - E_{n+1}^2)^2 + 4E_{n+2}^2 F_{n+1}^2\]

[The last equality follows from the fact that \(E_{n+1}^2 + E_{n}^2 = E_{2n+1}^2\) for each non-negative integer \(n\).]


**NOTE:** Each of Horadam and Shannon pointed out that both B-402 and B-426 are special cases of general equation (2.2)' in A.F. Horadam: "Special Properties of the Sequence \(\{a, b, p, q\}\)," *The Fibonacci Quarterly* 5 (1967):425.

**Closed Form, Ingeniously**

B-427 Proposed by Phil Mana, Albuquerque, NM

Establish a closed form for \(\sum_{k=1}^{n} k \binom{k}{2} \binom{n-k}{3}\).

Solution by Graham Lord, Université Laval, Québec, Canada

The sum \(\sum_{k=2}^{n-3} (k+1) \binom{k}{2} \binom{n-k}{3} = \sum_{k=2}^{n-1} (k+1) \binom{n-k}{3} \)

\[= 3 \cdot \sum_{k=2}^{n-3} \binom{k+1}{3} = \binom{n+1}{6}\]

(continued)
\[
= 3 \cdot \left( \frac{n + 2}{7} \right) - \left( \frac{n + 1}{6} \right) = \frac{3n - 1}{7} \cdot \left( \frac{n + 1}{6} \right).
\]

**NOTE:** As shown in my solution to B-425, \( \sum_{n}^{m} \binom{n}{a} \binom{m}{b} \) counts the number of subsets of \( a + b + 1 \) elements chosen from a set of \( n+1 \) elements: this latter sum equals \( \binom{n + 1}{a + b + 1} \).

Also solved by Paul S. Bruckman, Bob Prielipp, Sahib Singh, Gregory Wulczyn, and the proposer.

**Closed Form, Industriously**

**B-428** Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

For odd positive integers \( n \), establish a closed form for

\[
\sum_{k=0}^{2n+1} \binom{2n+1}{k} p_{n+k}^2.
\]

**Solution by Bob Prielipp, University of Wisconsin-Oshkosh, WI**

Since \( P_i = (a^i - b^i)/\sqrt{5} \) [where \( a = (1 + \sqrt{5})/2 \) and \( b = (1 - \sqrt{5})/2 \)],

\[
\sum_{k=0}^{2n+1} \binom{2n+1}{k} p_{n+k}^2 = \frac{1}{5} \sum_{k=0}^{2n+1} \binom{2n+1}{k} (a^{2n+2k} - 2(-1)^n a^{2k} + b^{2n+2k})
\]

\[
= \frac{1}{5} \left[ a^{2n} (1 + a^{2v})^{2n+1} - 2(-1)^n (1 + (-1)^v)^{2n+1} + b^{2n} (1 + b^{2v})^{2n+1} \right] \quad \text{(by the Binomial Theorem)}
\]

\[
= \frac{1}{5} \left[ a^{2n} (1 + a^{2v})^{2n+1} + b^{2n} (1 + b^{2v})^{2n+1} \right] \quad \text{(because \( v \) is odd)}
\]

\[
= \frac{1}{5} \left[ a^{2n+2(v+1)} (a^{-v} + a^v)^{2n+1} + b^{2n+2(v+1)} (b^{-v} + b^v)^{2n+1} \right]
\]

\[
= \frac{1}{5} \left[ a^{2n+2(v+1)} (a^{-v} - a^v)^{2n+1} + b^{2n+2(v+1)} (b^{-v} - b^v)^{2n+1} \right]
\]

\[
= \frac{1}{5} \left[ a^{2n+2(v+1)} (a^{-v} - b^v)^{2n+1} + b^{2n+2(v+1)} (b^{-v} - a^v)^{2n+1} \right]
\]

\[
= \frac{1}{5} \left[ a^{2n+2(v+1)} (a^{-v} + b^v)^{2n+1} + b^{2n+2(v+1)} (b^{-v} + a^v)^{2n+1} \right]
\]

\[
= \frac{1}{5} \left[ a^{2n+2(v+1)} (a^{-v} + b^v)^{2n+1} + b^{2n+2(v+1)} (b^{-v} + a^v)^{2n+1} \right]
\]

Also solved by Paul S. Bruckman, A. G. Shannon, and the proposer.

**Yes, When Boiled Down**

**B-429** Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

Is the function

\[
P_n^h - P_n^h - (L_{8r} + L_{4r} - 1)(P_{n+6r}^h + P_{n+2r}^h) + (L_{12r} - L_{8r} + 2)(P_{n+6r}^h + P_{n+4r}^h)
\]

independent of \( n \)? Here \( n \) and \( r \) are integers.

**Solution by Paul S. Bruckman, Concord, CA and Sahib Singh, Clarion State College, Clarion, PA, independently**
Yes. It boils down to

$$12\frac{P_2^2}{2r} \cdot \frac{p^2}{q_r}$$

or to

$$12(L_{12r} - 2L_{6r} - L_{4r} + 4)/25.$$  

(The steps were deleted by the Elementary Problems editor.)

Also solved by Bob Prielipp and the proposer.

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