FIBONACCI NUMBER IDENTITIES FROM ALGEBRAIC UNITS

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1. INTRODUCTION

In several recent papers L. Bernstein [1], [2] introduced a method of operating with units in cubic algebraic number fields to obtain combinatorial identities. In this paper we construct kth degree ($k \geq 2$) algebraic fields with the special property that certain units have Fibonacci numbers for coefficients. By operating with these units we will obtain our main result, an infinite class of identities for the Fibonacci numbers. The main result is given in Theorem 1 and illustrated in Figure 1.

2. MAIN RESULT

Theorem 1: For each posotive integer k let A_k be a $(2k - 1) \times (2k - 1)$ determinant, $A_k = \det(a_{ij})$, see Figure 1, where a_{ij} is given by

$$a_{ij} = \begin{cases} (-1)^{n+1}F_{n+2} & \text{if } i = j \text{ and } j < k \\ (-1)^n F_{n+1} + (-1)^{n+1}F_{n+2} & \text{if } i = j \text{ and } j \geq k \\ (-1)^n F_{n+1} & \text{if } i = j - k \text{ and } i < k \\ & \text{or } i = j + k \text{ and } i > k \\ 0 & \text{otherwise} \end{cases} \qquad (k > 1).$$

For $k = 1$, we define A_1 to be the middle entry in Figure 1, i.e.,

$$A_1 = F_{n+2} - F_{n+1}.$$

Then, for all $k \geq 1$, we have $F_n = |A_k|$.

$$A_k = \begin{array}{ccccccc} (-1)^{n+1}F_{n+2} & 0 \cdots\cdots 0 & (-1)^n F_{n+1} & 0 \cdots\cdots & & & 0 \\ 0 & \ddots & & & \ddots & & \vdots \\ \vdots & & (-1)^{n+1}F_{n+2} & & & & 0 \\ \vdots & & & \ddots & & & (-1)^n F_{n+1} \\ 0 & \cdots\cdots 0 & & (-1)^n F_{n+1} + (-1)^{n+1}F_{n+2} & 0 \cdots\cdots & & 0 \\ (-1)^n F_{n+1} & & & & & & \vdots \\ 0 & \ddots & & & \ddots & & \vdots \\ \vdots & & & & & & 0 \\ 0 & \cdots\cdots 0 & (-1)^n F_{n+1} & 0 \cdots\cdots 0 & & & (-1)^n F_{n+1} + (-1)^{n+1}F_{n+2} \end{array}$$

Fig. 1 (2k - 1) × (2k - 1) Determinant

Proof: Throughout the entire ensuing discussion, k will be a fixed positive integer. Consider the following $2k$ recursion formulas with the accompanying $2k$ initial conditions. For each fixed j, $j = 0, 1, \ldots, 2k - 1$, let

$$a_j(n + 2k) = a_j(n + k) + a_j(n) \qquad (n \geq 0) \tag{1}$$

and

$$a_j(n) = \begin{cases} 1 & \text{if } n = j \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In particular, for $k = j = 1$, we obtain

$$a_1(n + 2) = a_1(n + 1) + a_1(n) \qquad (n \geq 0)$$

and

$$a_1(0) = 0, \quad a_1(1) = 1,$$

that is, $\{a_1(n)\}_{n=1}^{\infty}$ is the Fibonacci sequence. In general, one can verify that for any fixed k and any j, $j = 0, 1, \ldots, 2k - 1$, the nonzero terms of the sequence $\{a_j(n)\}_{n=1}^{\infty}$ are the Fibonacci numbers. More precisely, from (1) and (2) one can obtain the equations:

$$\begin{aligned} a_j(k - 1 + kn) &= 0 \text{ if } j \neq 2k - 1 \text{ or } k - 1 \\ a_{k-1}(k - 1 + kn) &= F_{n-1} \\ a_{2k-1}(k - 1 + kn) &= F_n. \end{aligned} \tag{3}$$

Now consider the algebraic number field $Q(w)$ where $w^{2k} = 1 + w^k$. We claim that the nonnegative powers of w are given by the equation

$$w^n = a_0(n) + a_1(n)w + \cdots + a_{2k-1}(n)w^{2k-1}, \tag{4}$$

where the $a_j(n)$, $0 \leq j \leq 2k - 1$, satisfy (1) and (2). From (4) we obtain

$$\begin{aligned} w^{n+1} = a_{2k-1}(n) + a_0(n)w + a_1(n)w^2 + \cdots + (a_{k-1}(n) + a_{2k-1}(n))w^k \\ + \cdots + a_{2k-2}(n)w^{2k-1}. \end{aligned} \tag{5}$$

Comparison of the coefficients in (4) and (5) yields the following $2k$ equations:

$$\begin{aligned} a_0(n + 1) &= 0 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 1 \cdot a_{2k-1}(n) \\ a_1(n + 1) &= 1 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 0 \cdot a_{2k-1}(n) \\ a_2(n + 1) &= 0 \cdot a_0(n) + 1 \cdot a_1(n) + \cdots + 0 \cdot a_{2k-1}(n) \\ &\vdots \\ a_k(n + 1) &= 0 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 1 \cdot a_{k-1}(n) + \cdots + 1 \cdot a_{2k-1}(n) \\ &\vdots \\ a_{2k-1}(n + 1) &= 0 \cdot a_0(n) + 0 \cdot a_1(n) + \cdots + 1 \cdot a_{2k-2}(n) + 0 \cdot a_{2k-1}(n). \end{aligned} \tag{6}$$

This system of equations can be written more simply in matrix form as follows. Let C be the coefficient matrix of the $a_j(n)$. Explicitly, $C = (c_{ij})$ is a $(2k)$ by $(2k)$ matrix, where

$$\begin{aligned} c_{1,2k} &= 1 \\ c_{k+1,2k} &= 1 \\ c_{ij} &= 1 \text{ if } i = 1 + j \\ c_{ij} &= 0 \text{ otherwise.} \end{aligned}$$

Let T_n denote the following column matrix:

$$T_n = \begin{bmatrix} a_0(n) \\ \vdots \\ a_{2k-1}(n) \end{bmatrix} \qquad (n \geq 0) \tag{7}$$

The system (6) can now be written as

$$T_{n+1} = CT_n.$$

More generally, if I denotes the identity matrix, then

$$\begin{aligned} T_n &= IT_n \\ T_{n+1} &= CT_n \\ &\vdots \\ T_{n+2k} &= C^{2k}T_n. \end{aligned} \tag{8}$$

The characteristic equation of C is found to be

$$\det(C - \lambda I) = \lambda^{2k} - \lambda^k - 1 = 0. \tag{9}$$

The Hamilton-Cayley theorem states that every square matrix satisfies its characteristic equation. Hence,

$$C^{2k} - C^k - I = 0$$

$$(C^{2k} - C^k - I)T_n = 0,$$

and from (8)

$$T_{n+2k} = T_{n+k} + T_n. \tag{10}$$

From (7) and (10) we have

$$a_j(n + 2k) = a_j(n + k) + a_j(n), \; j = 0, \ldots, 2k - 1.$$

Thus (1) of our claim is established. The initial conditions for (10) can be obtained from (4) and are given by the $2k$ column matrices

$$T_j = (t_{i1}), \; j = 0, 1, \ldots, 2k - 1,$$

where

$$t_{i1} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \qquad i = 0, 1, \ldots, 2k - 1. \tag{11}$$

From (7) we have that $t_{i1} = a_i(n)$. Hence, $a_i(n) = 1$ if and only if $i = j = n$, and (2) is established, thus completing the proof of our claim.

From $w(w^{2k-1} - w^{k-1}) = 1$, we see that

$$w^{-1} = w^{2k-1} - w^{k-1}.$$

If we denote the negative powers of w by

$$w^{-n} = b_0(n) + b_1(n)w + \cdots + b_{2k-1}w^{2k-1} \qquad (n \geq 0), \tag{12}$$

then by calculations analogous to those used for the coefficients of the positive powers of w, we obtain the following results. The coefficients satisfy the recursion formulas,

$$b_j(n + 2k) = b_j(n) - b_j(n + k), \; j = 0, 1, \ldots, 2k - 1.$$

The initial conditions that are not zero are given by

$$\begin{aligned}
b_0(0) &= 1 \\
b_0(k) &= -1 \\
b_j(k-j) &= -1 \\
b_j(2k-j) &= 2 \qquad j = 1, 2, \ldots, k-1, \\
b_j(j) &= 1 \qquad j = k, \\
b_j(2k-j) &= 1 \\
b_j(3k-j) &= -1. \qquad j = k+1, \ldots, 2k-1,
\end{aligned} \tag{13}$$

The result analogous to (3) is given by

$$\begin{aligned}
b_1(k-1+kn) &= (-1)^{n+1}F_{n+2} \\
b_{k+1}(k-1+kn) &= (-1)^n F_{n+1} \\
b_j(k-1+kn) &= 0, \qquad \text{if } j \neq 1 \text{ or } k+1.
\end{aligned} \tag{14}$$

If we employ (4), (12), and (14), then omitting the argument $(k - 1 + kn)$ from the a_j and b_j, we can write

$$\begin{aligned}
1 &= w^{k-1+kn}w^{-(k-1+kn)} \\
&= (a_0 + a_1 w + \cdots + a_{2k-1}w^{2k-1})(b_1 w + b_{k+1}w^{k+1}).
\end{aligned}$$

Multiplying out the right-hand side and comparing coefficients, we obtain the $2k$ equations:

$$\begin{aligned}
a_{k-1}b_{k+1} + a_{2k-1}(b_1 + b_{k+1}) &= 1 \\
a_0 b_1 + a_k b_{k+1} &= 0 \\
a_1 b_1 + a_{k+1} b_{k+1} &= 0 \\
\vdots \qquad \vdots \qquad & \vdots \\
a_{k-2} b_1 + a_{2k-2} b_{k+1} &= 0 \\
a_{k-1}(b_1 + b_{k+1}) + a_{2k-1}(b_1 + 2b_{k+1}) &= 0 \\
a_0 b_{k+1} + a_k(b_1 + b_{k+1}) &= 0 \\
a_1 b_{k+1} + a_{k+1}(b_1 + b_{k+1}) &= 0 \\
\vdots \qquad \vdots \qquad & \vdots \\
a_{k-2} b_{k+1} + a_{2k-2}(b_1 + b_{k+1}) &= 0.
\end{aligned}$$

We will consider the $a_0, \ldots, a_{2k-1}$ as the unknowns and solve for a_{2k-1} by Cramer's rule. If we denote the coefficient matrix by D and use (3) and (14) to replace b_1, b_{k+1}, and a_{2k-1}, then Cramer's rule yields

$$F_n = \pm\frac{A_k}{\det D}.$$

We will complete the proof of the theorem by showing that $\det D = \pm 1$.

The norm of $e = b_1 w + b_{k+1}w^{k+1}$ is given by the determinant of the matrix whose entries are the coefficients of w^j, $j = 0, \ldots, 2k - 1$, in the following equations:

$$\begin{aligned}
e &= b_1 w + b_{k+1}w^{k+1} \\
ew &= b_1 w^2 + b_{k+1}w^{k+2} \\
\vdots \quad & \quad \vdots \qquad \vdots \\
ew^{k-1} &= b_{k+1} + (b_1 + b_{k+1})w^k
\end{aligned}$$

(continued)

$$ew^k = b_{k+1}w + (b_1 + b_{k+1})w^{k+1} \tag{15}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$ew^{2k-1} = b_1 + b_{k+1} + (b_1 + 2b_{k+1})w^k.$$

The norm of e is ± 1 since $e = w^{-(k-1+kn)}$ and w is a unit. We observe, however, that D is just the transpose of the matrix from which the norm of e was calculated. Hence, det $D = \pm 1$, and our theorem is proved.

As a concluding note we remark that, if $k = 2$, then the theorem yields—with the appropriate choice of the plus/minus signs—the identity

$$F_n = (-1)^{n+1}F_{n+2}^3 + 2(-1)^n F_{n+1}F_{n+2}^2 + (-1)^{n+1}F_{n+1}^3. \tag{16}$$

This can also be verified as follows: Replace F_{n+2} by $F_n + F_{n+1}$ in (16) and simplify to obtain

$$F_{n+1}^2 - F_{n+1}F_n - F_n^2 = (-1)^n. \tag{17}$$

Finally, compare (17) with the known [6, p. 57] identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

to complete the verification of (16).

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POINTS AT MUTUAL INTEGRAL DISTANCES IN S^n

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In radio-astronomy circles, it is sometimes jokingly speculated whether it is possible to place infinitely many in-phase, nonaligned antennas in a plane (say, vertical dipoles in a horizontal plane). Geometrically, this means placing infinitely many nonaligned points in R^2, with integral pairwise distances; and naturally the mathematician wants to generalize to R^3 and R^n. In R^3 there is still a physical meaning for acoustic radiators, but not for electromagnetic radiators, since none exists with a spherical symmetry radiation pattern (for more serious questions on antenna configurations, see [2]).

A slightly different problem is that of placing a receiving antenna in a point P, where it receives in phase from transmitting antennas placed in nonaligned coplanar points A_1, A_2, ... (in phase with each other); geometrically,