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THE UBIQUITOUS RATIONAL SEQUENCE

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DEDICATION

Vern Hoggatt has been the inspiration of many papers that have appeared in this journal. He shared his enthusiasm and curiosity about mathematics with a notable generosity. His students, friends, and pen pals were enriched by the problems he posed and often helped to solve. My own interest in sequences was greatly influenced by the correspondence we started when I was a graduate student at the University of Alberta. Some of my first papers written at that *time were solutions to research problems he posed. To this day, rational sequences permeate my research work. It seems appropriate to present my own view of the theory of finite differences which has evolved over the years. This paper will be useful for the beginner, the sort of person Vern Hoggatt helped so much, and it should have some novelty for others as well. In it, I hope to show how rational sequences fit into some parts of mathematics—linear algebra and elementary calculus in particular. The exposition will be brief with plenty of gaps to be filled in by the reader.*

1. RATIONAL SEQUENCES

What is a rational sequence? A mapping f from $\mathbb{N} = \{0, 1, 2, \ldots\}$ into a field $\mathfrak{F}$ is *rational* if and only if there exist elements $c_1, \ldots, c_k \in \mathfrak{F}$ with $c_k \neq 0$, and there exists $h \in \mathbb{N}$ with $k \leq h$ such that

$$(1) \qquad f(n) = c_1 f(n-1) + \cdots + c_k f(n-k) \qquad (n \in \mathbb{N},\ h < n).$$

Sometimes a rational sequence is said to satisfy a linear homogeneous difference equation with constant coefficients. This long phrase is usually shortened to "difference equation" or "linear recurrence." We refer to (1) as the *difference equation form*, meaning it is one way of presenting a rational sequence. The term "rational" is short, and it describes a characteristic feature of such sequences. Namely, the generating function of f is rational (the quotient of two polynomials); in fact, the generating function is

$$(2) \quad \sum_{n=0}^{\infty} f(n) z^n = \frac{f(0) + \{f(1) - c_1 f(0)\}z + \cdots + \{f(h) - \cdots - c_k f(h-k)\}z^h}{1 - c_1 z - \cdots - c_k z^k}.$$

We refer to (2) as the *generating function form* of the definition of f. For example, the difference equation form of the Fibonacci sequence is

$$F_0 = 0,\ F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for all } n \geq 2.$$

This is equivalent to the generating function form

$$(3) \qquad \sum_{n=0}^{\infty} F_n z^n = \frac{z}{1 - z - z^2}.$$

Perhaps it should be emphasized that (2) and (3) have purely algebraic interpretations. We are merely using the formal sum as a convenient notation for a sequence. For example, (3) only means that the Cauchy product of the sequences $(1, -1, -1, 0, 0, \ldots)$ and $(F_0, F_1, F_2, \ldots)$ is equal to $(0, 1, 0, 0, \ldots)$. In terms of formal power series, this means

$$(4) \qquad (1 - z - z^2) \sum_{n=0}^{\infty} F_n z^n = z.$$

We are not concerned with the fact that the power series on the left-hand side in (3) represents the rational function on the right-hand side for certain values of z. Such a discussion would have to be given to justify putting $z = \frac{1}{2}$ in (3) to conclude

$$(5) \qquad \frac{F_1}{2} + \frac{F_2}{4} + \cdots + \frac{F_n}{2^n} + \cdots = 2,$$

but this is not the sort of application we have in mind. The algebraic basis can be found in [1], for example.

Rational sequences may be recognized as such in other ways than by the difference equation or rational generating function. Next most important after these is the exponential form. To get to the heart of the matter, suppose $P(z)/Q(z)$ is the generating function of the rational sequence f with P, Q polynomials over $\mathfrak{F}$ such that $Q(0) = 1$, and P, Q have no common zeroes. (That is, P/Q is a "reduced fraction.") Also, there is no loss in generality to assume P has degree less than that of Q. (Otherwise, write $P/Q = R + S/Q$ where R, S are polynomials with the degree of S less than that of Q.) Also, it can be supposed that the zeroes of Q are elements of $\mathfrak{F}$ (otherwise, just extend $\mathfrak{F}$ by these zeroes). Suppose the distinct zeroes of Q are $1/\theta_1, \ldots, 1/\theta_t$ [$\theta_1, \ldots, \theta_t \neq 0$ because $Q(0) = 1$], and let d_i denote the multiplicity of $1/\theta_i$ for $i = 1, \ldots, t$. Then $Q(z) = (1 - \theta_1 z)^{d_1} \cdots (1 - \theta_t z)^{d_t}$, and it can be shown that there exist polynomials $P_1, \ldots, P_t$ over $\mathfrak{F}$ with the degree of P_i less than d_i for $i = 1, \ldots, t$ such that

$$\frac{P(z)}{Q(z)} = \frac{P_1(z)}{(1 - \theta_1 z)^{d_1}} + \cdots + \frac{P_t(z)}{(1 - \theta_t z)^{d_t}}. \tag{6}$$

Rather than give an explicit formula for the coefficients of $P_i(z)$, we will just show how to compute them. To do this, it is enough to consider the case $i = 1$. Start with

$$P_1(z) + (1 - \theta_1 z)^{d_1} \sum_{i=2}^{t} \frac{P_i(z)}{(1 - \theta_i z)^{d_i}} = \frac{P(z)}{\sum\limits_{i=2}^{t} (1 - \theta_i z)^{d_i}}, \tag{7}$$

and differentiate $d_1 - 1$ times with respect to z to obtain d_1 equations involving the various derivatives of the polynomial $P_1(z)$. Then put $z = 1/\theta_1$ in each of these equations to get

$$D^j\left\{P_1(z)\right\}_{z=1/\theta_1} = D^j\left\{P(z)\sum_{i=2}^{t}(1 - \theta_2 z)^{-d_i}\right\}_{z=1/\theta_1} \qquad (j = 0, \ldots, d_1 - 1), \tag{8}$$

where D denotes the differential with respect to z. (All of this can be done in an algebraic manner by introducing a formal operation on sequences; calculus is not actually required.) Note that by putting $z = 1/\theta_1$ in the jth differential equation, all of the terms involving $P_2, \ldots, P_t$ have a factor $(1 - \theta_1 z)$, so these terms drop out of the computation. This gives rise to a linear system of d_1 equations in the d_1 coefficients of P_1. This system can be solved because the matrix of the system is upper triangular and has a nonzero diagonal. Once we have $P_1, \ldots, P_t$ in (6), we can develop each of the t rational functions on the right into a power series using the binomial theorem. In fact, the full force of the binomial theorem is not needed. One only needs

$$\frac{1}{(1 - z)^d} = \sum_{n=0}^{\infty} \binom{n + d - 1}{d - 1} z^n, \tag{9}$$

and this can be established by induction on d. Thus, if

$$P_i(z) = p_0 + p_1 z + \cdots + p_{d_i - 1} z^{d_i - 1},$$

then

$$\frac{P_i(z)}{(1 - \theta_i z)} = \sum_{n=0}^{\infty} \left\{ p_0 \binom{n + d_i - 1}{d_i - 1} + \frac{p_1}{\theta_i}\binom{n + d_i - 2}{d_i - 1} + \cdots \right\} \theta_i^n z^n \tag{10}$$

for $i = 1, \ldots, t$. Since $\binom{n + d - 1}{d - 1}$ is a polynomial in n with degree $d - 1$, the coefficient of z^n in the right member of (10) has the form $\pi_i(n)\theta_i^n$, where

$\pi_i(n)$ is a polynomial in n whose degree is $d_i - 1$. Summing over i, we can conclude that

$$f(n) = \pi_1(n)\theta_1^n + \cdots + \pi_t(n)\theta_t^n \quad (n \in \mathbb{N}), \tag{11}$$

where $1/\theta_1, \ldots, 1/\theta_t$ are the distinct zeroes of $Q(z)$ with multiplicities $d_1, \ldots, d_t$, respectively, and $\pi_1, \ldots, \pi_t$ are polynomials over $\mathfrak{F}$ where π_i has degree less than d_i for $i = 1, \ldots, t$. We call (11) the *exponential form* for the rational sequence f. We derived the exponential form from the rational form, but it is important to note that given any one of the forms (1), (2), or (11), the other two can be derived from it.

Continuing the example dealing with the Fibonacci sequence, note that $1 - z - z^2$ has zeroes $1/\alpha$, $1/\beta$ where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$. Hence

$$1 - z - z^2 = (1 - \alpha z)(1 - \beta z),$$

and using the method outlined above, we find

$$\sum_{n=0}^{\infty} F_n z^n = \frac{z}{1 - z - z^2} = \frac{\alpha z}{1 - \alpha z} - \frac{\beta z}{1 - \beta z} = \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} z^n. \tag{12}$$

Thus, the Fibonacci sequence has the exponential form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (n \in \mathbb{N}), \tag{13}$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$.

There is still another useful presentation of a rational sequence called the *matrix form*. Let

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ c_k & c_{k-1} & c_{k-2} & c_{k-3} & \cdots & c_1 \end{bmatrix}, \quad v_n = \begin{bmatrix} f(n) \\ f(n+1) \\ \vdots \\ f(n+k-1) \end{bmatrix} \quad (n \in \mathbb{N}), \tag{14}$$

where M is a $k \times k$ matrix having $[c_k, \ldots, c_1]$ as its bottom row and the $(k - 1) \times (k - 1)$ identity matrix as the minor of the $(k, 1)$-entry. It is easy to verify that $Mv_n = v_{n+1}$ for all $n \in \mathbb{N}$, and hence that $M^n v_0 = v_n$ for all $n \in \mathbb{N}$. Since M^n can be computed in about $\log n$ matrix multiplications, it follows that $f(n)$ can be computed in $O(\log n)$ basic steps instead of the $O(n)$ steps one might guess. Note that the eigenvalues of M are $\theta_1, \ldots, \theta_t$ with multiplicities $d_1, \ldots, d_t$, respectively, because the characteristic polynomial of M is

$$\det(M - zI) = (-z)^k Q(1/z) = (-1)^k (z^k - c_1 z^{k-1} - \cdots - c_k).$$

(This can be shown by a simple induction proof on k.)

In the case of the Fibonacci sequence, we have

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad v_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} \quad (n \in \mathbb{N}). \tag{15}$$

Hence

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} \quad (n \in \mathbb{N}). \tag{16}$$

The difference equation

$$x_n = c_1 x_{n-1} + \cdots + c_k x^{n-k} \tag{17}$$

with $c_1, \ldots, c_k \in \mathfrak{F}$, $c_k \neq 0$, has *order* k. The *order of a rational sequence* is the minimum order of all difference equations it satisfies. A rational sequence f of order k satisfies a unique difference equation of order k. [The uniqueness depends on the standard form given in (17); after all, nothing is changed by multiplying through (17) with a nonzero element of $\mathfrak{F}$.] In general, a rational sequence of order k satisfies as many difference equations of order $k + d$ as there are polynomials R over $\mathfrak{F}$ with degree d and $R(0) = 1$. To see that the difference equation of lowest order satisfied by f is unique, suppose for the moment there are two. Say f satisfies (1) and

$$f(n) = b_1 f(n-1) + \cdots + b_k f(n-k) \qquad (h \leq n). \tag{18}$$

Taking the difference of equations (1) and (18) leads to a new difference equation with order less than k satisfied by f if $b \neq c$ for some i with $1 \leq i \leq k$. So k cannot be the order of f as was assumed. If f has order k and (17) is the unique difference equation of order k satisfied by f (this is called the *minimal equation*), then the generating function of f has the form (2). Let P and Q denote the numerator and denominator, respectively, in the right member of (2), and note that $Q(0) = 1$, and P and Q have no common zeroes. (Otherwise, g would satisfy a difference equation of order $k - 1$.) We call the rational function P/Q the *canonical generating function* of f, and note that it is unique. For each polynomial R with degree d over $\mathfrak{F}$ with $R(0) = 1$, we have $P/Q = PR/QR$, so f satisfies a difference equation of order $k + d$ with coefficients equal to the coefficients of QR. All difference equations of order $k + d$ satisfied by f are obtained in this way, because each difference equation of order $k + d$ satisfied by f gives rise to polynomials U, V over $\mathfrak{F}$ with $V(0) = 1$ such that $P/Q = U/V$. But this means $PR = U$ and $QR = V$ with R a polynomial over $\mathfrak{F}$ with degree d and $R(0) = 1$. We conclude this discussion of the order of a sequence by observing that the order of f can be deduced from its exponential form by adding t to the sum of the degrees of $\pi_1, \ldots, \pi_t$.

2. SITUATIONS IN WHICH RATIONAL SEQUENCES ARISE

Sometimes rational sequences are formed in terms of other rational sequences. For example, if f, g are rational sequences over the field $\mathfrak{F}$ and $a, b \in \mathfrak{F}$, then we can form a new sequence $h = af + bg$ defined by

$$h(n) = af(n) + bg(n) \qquad (n \in \mathbb{N}). \tag{19}$$

Let F, G, H denote the generating functions of F, G, H, respectively, then $H = aF + bG$. This means that H is a rational function because F and G are, so h is a rational sequence. It is easy to check that the set of all rational sequences over $\mathfrak{F}$ forms a subspace of the vector space of all sequences over $\mathfrak{F}$. Furthermore, the sequences which satisfy equation (17) form a k-dimensional subspace. If $\theta_1, \ldots, \theta_t$ denote the zeroes of $z^k - c_1^{k-1} - \cdots - c_k$ with multiplicities $d_1, \ldots, d_t$, respectively, it is easy to check that each of the sequences

$$(n^j \theta_i^n : n \in \mathbb{N}) \text{ for all } j \in \mathbb{N} \text{ and } i = 1, \ldots, t$$

satisfies (17). Using the exponential form for any sequence f which satisfies (17), it follows that the k sequences

$$(n^j \theta_i^n : n \in \mathbb{N}) \qquad 0 \leq j < d, \; i = 1, \ldots, t \tag{20}$$

form a basis for the vector space of all sequences which satisfy (17). That this actually is a basis depends heavily on the proof that every solution of (17) has the exponential form given in (11).

There are ways other than forming linear combinations to build new rational sequences from those on hand. For example, consider the Cauchy product $f \times g$

or the termwise product $f \cdot g$ defined, respectively,

(21) $$(f \times g)(n) = \sum_{i=0}^{n} f(i)g(n-i) \qquad (n \in \mathbb{N}),$$

(22) $$(f \cdot g)(n) = f(n)g(n) \qquad (n \in \mathbb{N}).$$

Let $h = f \times g$, and let F, G, H denote the generating functions of f, g, h, respectively. Then $H = FG$, and H is a rational function if the same is true of F and G. Hence, h is a rational sequence if f and g are. To see that $f \cdot g$ is rational whenever f and g are, use the exponential form of f and g. It is fairly easy to check that the product of two exponential forms is again an exponential form, so this approach gives a proof. The generating function of $f \cdot g$ can be given in terms of F and G by means of a contour integral as was shown in [2]. The fact that the termwise product of two rational sequences is again rational seems to be due to Vaidyanathaswamy [10].

The termwise product can be used to produce all sorts of unexpected results. For example, since the Fibonacci sequence is rational, it follows that

$$(F_n^j : n \in \mathbb{N})$$

is rational for all $j \in \mathbb{P}$. The minimal equation for the jth powers of the Fibonacci sequence were given in [9]. Also, the sequence p defined by $p(n) = n$ $(n \in \mathbb{N})$ satisfies

$$p(n) = 2p(n-1) - p(n-2),\ 2 \le n,$$

so $p^j = (n^j : n \in \mathbb{N})$ is rational for all $j \in \mathbb{P}$. Hence, the linear combination $q = a_0 + a_1 p + \cdots + a_j p^j$ is also rational, and so is $g \cdot f$ for any rational f. For example, again using the Fibonacci sequence,

$$(n^2 F_n - n + 2 : n \in \mathbb{N})$$

is rational. A little subtler use of the termwise product involves periodic sequences. Suppose s is a sequence such that $s(n) = s(n-m)$ for all $n \ge h$ for some h, $m \in \mathbb{N}$; that is, s is *eventually periodic* and has *period* m. By definition, s satisfies a difference equation, so s is rational. In particular, let a, $m \in \mathbb{N}$ with $0 < m$, and define $s(n) = 1$ whenever $a \le n$, $n \equiv a \pmod m$, and $s(n) = 0$ otherwise. Since s is eventually periodic, $s \cdot f$ is rational whenever f is; furthermore, the generating function of $s \cdot f$ has the form $z^a P(z^m)/Q(z^m)$ with P, Q polynomials over $\mathfrak{F}$ and $Q(0) = 1$. Hence, the sequence g defined by $g(n) = f(mn + a)$ for all $n \in \mathbb{N}$ has $P(z)/Q(z)$ as its generating function, so g is rational. For example, the subsequence $(F_2, F_7, F_{12}, \ldots) = (F_{5n+2} : n \in \mathbb{N})$ of the Fibonacci sequence is rational (the difference equation is

$$x_n = 11x_{n-1} + x_{n-2},\ 2 \le n).$$

Interwoven rational sequences are also rational. More precisely, suppose f_0, $\ldots$, f_{m-1} are rational sequences, and define f by

(23) $$f(n) = f_r(n) \qquad [\text{where } n \equiv r \pmod m,\ n \in \mathbb{N}].$$

Let F, F_0, $\ldots$, F_{m-1} denote the generating functions of f, f_0, $\ldots$, f_{m-1}, respectively, then

(24) $$F(z) = F_0(z^m) + zF_1(z^m) + \cdots + z^{m-1}F_{m-1}(z^m).$$

Since F_0, $\ldots$, F_{m-1} are rational functions, so is F; therefore, f is a rational sequence.

Sometimes a finite set of sequences is defined by means of some initial conditions and a finite set of difference equations. It turns out that each of the sequences is rational in this case. This can be formulated more precisely as follows: Let f_1, $\ldots$, f_m be sequences, and suppose for each i, $1 \le i \le m$, there exists $h_i \in \mathbb{N}$, together with finite sets $S_{i1}, \ldots, S_{im} \subseteq \mathbb{N}$ and constants c_{ijk} corresponding to each $j \in S_{ik}$ such that

$$f_i(n) = \sum_{k=1}^{m} \sum_{j \in S_{ik}} c_{ijk} f_k(n - j) \qquad (n \in \mathbb{N},\ h_i < n) \tag{25}$$

for $i = 1, \ldots, m$. Also, suppose $f_i(n)$ is given for all n with $n \leq h_i$ for $i = 1, \ldots, m$, and suppose that this boundary condition together with the system (25) gives an unambiguous algorithm to compute the sequences $f_1, \ldots, f_m$. Then each of $f_1, \ldots, f_m$ is rational. To see this, convert the system (25) to a system of linear equations in the generating functions $F_1, \ldots, F_m$. The coefficients in this system are polynomials in z over the field $\mathfrak{F}$. This system can be solved using Cramer's Rule to deduce that each of $F_1, \ldots, F_m$ is a rational function. In fact, F_i has the form P_i/Q where Q is the determinant of the system, and P_i is a polynomial computed in a similar fashion.

A particular case of the foregoing situation involves matrices. Suppose $M = [c_{ij}]$ is an $m \times m$ matrix over the field $\mathfrak{F}$, and let $v_0 = [f_1(0), \ldots, f_m(0)]^T$ (where T denotes the transpose operator). Define v_n for all $n \in \mathbb{N}$ by $v_{n+1} = Mv_n$. This is equivalent to the system of difference equations

$$f_i(n + 1) = c_{i1}f_1(n) + \cdots + c_{im}f_m(n) \qquad (n \in \mathbb{N}) \tag{26}$$

for $i = 1, \ldots, m$. In terms of generating functions, this becomes

$$MF = v_0 \tag{27}$$

where $F = [F_1, \ldots, F_m]$. The determinant of this system is the characteristic polynomial of M; that is, $\det(M - zI)$. This gives information about the denominator polynomials in the generating functions $F_1, \ldots, F_m$. This observation can be taken a little further to deduce the Cayley-Hamilton Theorem as was done in [3].

One might get the impression that the rational sequence f_1 (defined in the previous paragraph) has order m, and that the minimal equation is given by the characteristic polynomial of M. But this is not always the case, and then Krylov's method may be useful. (See [11].) The idea here is to look for a linear dependency among the vectors $M^0v_0, M^1v_0, \ldots, M^kv_0$ for $k = 1, 2, \ldots$. Once one has $c_0, \ldots, c_k \in \mathfrak{F}$ for some minimal k such that

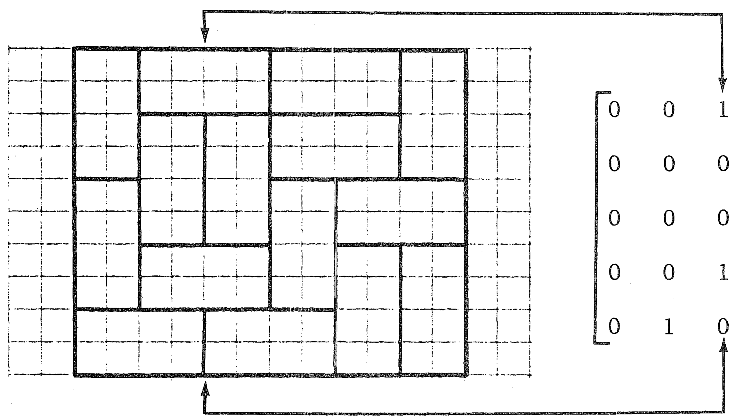
$$c_0M^0v_0 + \cdots + c_kM^kv_0 = M^{k+1}v_0, \tag{28}$$

multiply through (28) with M^n to deduce that f_1 satisfies

$$c_0x_n + c_1x_{n+1} + \cdots + c_kx_{n+k} = x_{n+k+1} \qquad (n \in \mathbb{N}). \tag{29}$$

3. SOME APPLICATIONS

This section gives brief descriptions of some recent results obtained by the author which involve rational functions. We start with domino tilings of rectangles with fixed width [4]. The idea here is based on an old, well-known observation about the number of paths of fixed length in a directed graph. Let $V = \{1, \ldots, m\}$, let $E \subseteq V \times V$, and let $M = [e_{ij}]$ be an $m \times m$ matrix defined by $e_{ij} = 1$ if $(i, j) \in E$ and $e_{ij} = 0$ otherwise. Elements of V are *vertices*, elements of E are *directed edges*, and M is the matrix of the *directed graph* (V, E). A sequence $(v_0, \ldots, v_k)$ is a *path of length* k in (V, E) just when $(v_{i-1}, v_i) \in E$ for $i = 1, \ldots, k$. It is well known that the number of paths $(v_0, \ldots, v_k)$ of length k in (V, E) with $v_0 = i$ and $v_k = j$ is the (i, j)-entry in M^k. Suppose we are only interested in paths which begin and end with vertex 1. Then let $c^{(k)}$ denote the first column of M^k for all $k \in \mathbb{N}$, and observe that $Mc^{(k)} = c^{(k+1)}$ for all $k \in \mathbb{N}$. We want the top element $c_{11}^{(k)}$ of $c^{(k)}$ for all $k \in \mathbb{N}$, so the method outlined in the last paragraphs of Section 2 can be applied. In particular, it follows that $(c_{11}^{(k)} : k \in \mathbb{N})$ is rational, and Krylov's method can be used to find a difference equation. Now let us see how this applies to domino tilings. Let $t(m, n)$ denote the number of tilings of an $m \times n$ rectangle with dominoes for all $m, n \in \mathbb{N}$. We fix the width m and concentrate on the computation of the sequence $(t(m, n) : n \in \mathbb{N})$. To do this, we create a graph whose vertices are cross-sections of tilings, and two cross-sections form a directed edge in the graph just when one can immediately follow the other in some tiling. A cross-section is a grid line parallel to the end of width m which cuts across some dominoes and passes others. Cross-sections can be encoded as binary sequences: 1 denotes a cut domino, and 0 denotes a crack between. For example, the 5 x 6 tiling shown in Figure 1 is encoded by the columns of the 0-1 matrix shown to its right. If we make the all-zero cross-section vertex 1, the $m \times n$ domino tilings correspond one-to-one with paths of length n beginning and ending at vertex 1. More details can be found in [4].



$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Fig. 1. A 5 x 6 domino tiling with its binary cross-section encoding

Now we give an example illustrating how a rational sequence can arise in a system of difference equations. Let A denote a finite set called an *alphabet*, and let A^* denote the set of all finite sequences of elements of A. Such sequences are called *words*, and in particular Λ denotes the empty word. Let F denote a finite subset of A^* and let A^*/F denote the set of all elements of A^* which do not have any elements of F as subwords. Elements of A^* belonging to A^*/F are called *good* and others are called *bad*. Let w denote a weight function defined on A^* such that $w(uv) = w(u)w(v)$ for all $u, v \in A^*$. Suppose further that for each $u \in A^*/F$ the sum

$$G_u = \sum_{(uv \in A^*/F)} w(uv)$$

is also a weight. The problem is to compute

$$G = G_\Lambda = \sum_{(u \in A^*/F)} w(u).$$

It was shown in [5] that G is a rational function in the weights of elements of A. This follows from two equations:

$$G = w(\Lambda) + \sum_{a \in A} Ga, \tag{30}$$

and

$$G_u = w(u)\left\{G - \sum G_v\right\}, \tag{31}$$

where the sum in the right member of (31) is over all basic words uv with $v \in A^*/F$. A word is *basic* if it is bad but no proper initial subword is bad. Note that if u is good, and uv is basic, then v is not longer than n where $n + 1$ is the length of the longest word in F. (A terminal subword of uv is an element of F and must overlap u.) Together (30) and (31) give rise to a linear system involving G_u for all good words u not longer than n. A procedure may be followed to keep this system small. First, write down (30). Then in subsequent stages write down expressions for those G_u which have appeared on the right side of earlier expressions obtained from (31). Since the length of u is bounded by n, this procedure terminates leaving us with a system linear in certain G_u, $u \in A^*/F$. We may conclude from the general argument given in Section 2 that G_u is rational in the weights $w(a)$, $a \in A$; in particular, this is true of G.

The result just described was used in [5] to treat a special case of the following unsolved problem. Let $\alpha_i(x) = m_i x + a_i$ be an affine function defined on the integers with $m_i, a_i \in \mathbb{N}$, $m_i > 1$, for $i = 1, \ldots, k$. Let $\langle A \rangle$ denote the semigroup generated by $A = \{\alpha_1, \ldots, \alpha_k\}$ under composition of functions. Note that an element $\alpha \in \langle A \rangle$ has the form $\alpha(x) = mx + a$ with m a product of the numbers $m_1, \ldots, m_k$. Let $p_1, \ldots, p_h$ denote the distinct prime divisors of $m_1, \ldots, m_k$, and for each $\alpha \in \langle A \rangle$ with $\alpha(x) = p_1^{i_1} \ldots p_h^{i_h} x + a$, let $w(\alpha) = x_1^{i_1} \ldots x_h^{i_h}$. It is easy to check that $w(\alpha\beta) = w(\alpha)w(\beta)$ for all $\alpha, \beta \in \langle A \rangle$ where $\alpha\beta(x) = \alpha(\beta(x))$. Is it true that

$$\sum_{u \in \langle A \rangle} w(u)$$

is a rational function? This problem has been solved when $m_i = m^{e_i}$ for some e_i, $m \in \mathbb{Z}$, $i = 1, \ldots, k$; the case when $e_i = \cdots = e_k = 1$ is treated in [6], and the systems of difference equations play an important role.

We conclude with an example which illustrates a frequently used formula from combinatorics. Let A denote a finite alphabet, let A^* denote the set of words over A, and let w denote a weight function on A^* which satisfies $w(uv) = w(u)w(v)$ for all $u, v \in A^*$. Then

$$\sum_{u \in A^*} w(u) = \frac{1}{1 - \sum_{a \in A} w(a)}. \tag{32}$$

Thus, rational functions arise. For example, this simple formula together with the inclusion-exclusion formula were used in [7] and [8] to show that the sequence of forms assumed by growing crystals is rational. more precisely, let $H, D \in \mathbb{Z}^k$ be finite sets, and consider the sequence of *crystals*

$$H,\ H + D,\ H + D + D,\ \ldots$$

formed by starting with the initial hub H, and adding increments equal to D in subsequent stages. Such a sequence is indicated in Figure 2 with $k = 2$, $H = \{(0, 0)\}$, $D = \{(0, 0), (1, 0), (0, 1)\}$.

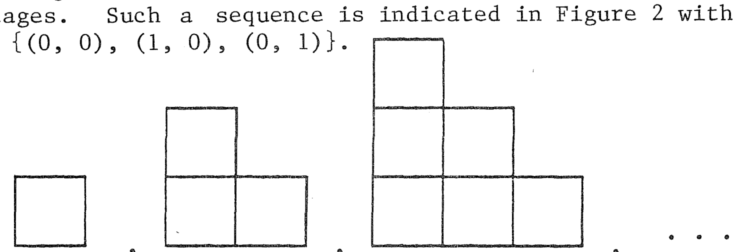


Fig. 2. A growing crystal

Give an element $i = (i_1, \ldots, i_k) \in \mathbb{Z}^k$ weight $w(i) = x_1^{i_1} \ldots x_k^{i_k}$ and define the weight $w(S)$ of $S \subseteq \mathbb{Z}^k$ to be the sum of the weights of the elements of S. The main result is that

$$w(H) + w(H + D)z + w(H + D + D)z^2 + \cdots \tag{33}$$

is a rational function in $x_1, \ldots, x_k$ and z. A consequence of this is that the sequence of volumes $(|H|, |H + D|, |H + D + D|, \ldots)$ forms a rational sequence.

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ON THE PROBABILITY THAT n AND $\Omega(n)$ ARE RELATIVELY PRIME

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To the memory of V. E. Hoggatt Jr.—my teacher and friend

It is a well-known result due to Chebychev that if n and m are randomly chosen positive integers, then $(n, m) = 1$ with probability $6/\pi^2$. It is the purpose of this note to show that if $\Omega(n)$ is the number of prime factors of n counted with multiplicity, then the probability that $(n, \Omega(n)) = 1$ is also $6/\pi^2$. Thus, as far as common factors are concerned, $\Omega(n)$ behaves randomly with respect to n.

Results of this type for fairly general additive functions have been proved by Hall [2], and in [1] and [3] he looks closely at the situation regarding the special additive function $g(n)$, the sum of the distinct prime factors of n. Hall's results do not apply to either $\Omega(n)$ or $\omega(n)$, the number of distinct prime factors of n, and so our result is of interest. Our proof, which is of an