3. Now we are in a position to prove (1.3) and (1.4). We have

$$S(2m,\ m) = \frac{1}{5}(2^{2m} + 2\alpha^{2m} + 2\beta^{2m}),$$

$$S(2m,\ m - 2) = \frac{1}{5}\left(2^{2m} + 2\alpha^{2m} \cos\frac{4\pi}{5} + 2\beta^{2m} \cos\frac{2\pi}{5}\right),$$

so

$$\begin{aligned} S(2m,\ m) - S(2m,\ m - 2) &= \frac{2}{5}\alpha^{2m}\left(1 - \cos\frac{4\pi}{5}\right) + \frac{2}{5}\beta^{2m}\left(1 - \cos\frac{2\pi}{5}\right) \\ &= \frac{2}{5}\alpha^{2m} \cdot \frac{\sqrt{5}}{2}\alpha + \frac{2}{5}\beta^{2m} \cdot \frac{\sqrt{5}}{2}(-\beta) \\ &= \frac{1}{\sqrt{5}}(\alpha^{2m+1} - \beta^{2m+1}) \\ &= F_{2m+1}, \end{aligned}$$

which is (1.3a). The derivations of (1.3b) and (1.4) from (2.4) are similar, and are omitted.

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SOME CONSTRAINTS ON FERMAT'S LAST THEOREM

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1. INTRODUCTION

The proof of "Fermat's Last Theorem," namely that there are no nontrivial integer solutions of $x^n + y^n = z^n$, where n is an integer greater than 2, is well known for the cases $n = 3$ and 4. We propose to look at some constraints on the values of x, y, and z, if they exist, when $n = p$, an odd prime. The history of the extension of the bounds on is interesting and illuminating [3], as is the development of the theory of ideals from Kummer's attempt to verify Fermat's result for all primes [2].

2. CONSTRAINT ON z

It can be readily established that there is no loss of generality in assuming that $0 < x < y < z$. Since $x \neq y$, $z - y \geq 1$ and $z - x \geq 2$. Following Guillotte [4], we consider $(x/z)^i + (y/z)^i = 1 + e_i$, where $e_0 = 1$, $e_p = 0$, and $e_i \in (0, 1)$ for $1 \leq i \leq p$. Summing over i from 0 to p, Guillotte further showed that

$$1/(1 - x/z) + 1/(1 - y/z) > p + 1 + \sum_{i=0}^{p} e_i,$$

from which we obtain

$$z(1/(z - x) + 1/(z - y)) > p + 2.$$

Since
$$z(1/(z - x) + 1/(z - y)) \leq z(\tfrac{1}{2} + 1);$$
$$3z/2 \geq z/(z - x) + z/(z - y) > p + 2.$$
Hence
$$z > 2(p + 2)/3.$$
Thus, if solutions in integers exist for the case when $p = 7$, we must have $z > 6$.

3. CONSTRAINT ON x

Now let $hx + y = z$. Since $z - y \geq 1$, $x \geq 1/h$. It has been shown in [1] that $h < 2^{1/p} - 1$, and so
$$x > 1/(2^{1/p} - 1).$$
Hence, if integer solutions exist for $p = 7$, we know that $x > 9.607$. Since $z \geq x + 2$, we know for $p = 7$ that $z \geq 11.607$, which is better than the bound found in Section 2.

4. CONSTRAINT ON y

Since $1/x < 2^{1/p} - 1$, we have $1 + p/x < 2$. Hence, $x > p$. For the case $p = 7$ we have that, if solutions exist, then $x > 7$, which is not an improvement on the result in Section 3. However, from [1], $z \to y + 1$ as $p \to \infty$, and thus if solutions in integers exist for very large values of p, then very large values of x and y are involved. We note also, since
$$2^{1/p} = \sum_{r=0}^{\infty}\left(\frac{1}{p} \ln 2\right)^r/r!,$$
that
$$2^{1/p} - 1 < \ln 2/(p - \ln 2),$$
which with the results from Sections 2 and 3 gives
$$y > p/\ln 2.$$
When $p = 7$, this yields $x > 9.099$ compared with $x > 9.607$ from Section 3. However, as p increases, the inequalities become closer, and the simpler $y > p/\ln 2$ is adequate. $y > 1.442695$ is also "sharper" than $x > p$.

Zeitlin [6] proved that no integer solutions exist for $x + ny \leq nz$. We note that for $n = 7$, $x > 9.607$, $y > 10.099$, $z \geq 11.607$ as above, $x + ny > 80.300$ and $nz \geq 81.249$. Perisastri [5] showed that in our notation
$$\sqrt{2x} > \sqrt{y}(1 + 1/(2p \ln 2p)).$$
For $p = 7$ and x, y, z as above, $\sqrt{2x} = 13.586$ and $\sqrt{y}(1 + 1/(2p \ln 2p)) = 3.264$.

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