Conjecture: If x_k, y_k is a node for $1 \leq k \leq F_N - 1$ and if N is $\binom{\text{odd}}{\text{even}}$, then $\binom{y_k,\ 1 - x_k}{y_k,\ x_k}$ is also a node.

Perhaps a reader can supply a proof.

One would expect the nodes of an efficient cubature rule to be symmetric about the center of the square so as to give identical results for $f(x, y)$, $f(x, 1 - y)$, $f(1 - x, y)$, and $f(1 - x, 1 - y)$. This suggests modifying (1) to

$$\int_0^1 \int_0^1 f(x, y)\, dx dy = \frac{f(0, 0) + f(0, 1) + \sum_{k=1}^{F_N} f(x_k, y_k) + f(x_k, 1 - y_k)}{2(F_N + 1)}. \tag{2}$$

Essentially, we have completed the square on the nodes. Some preliminary calculations* indicated that this gain in accuracy more than compensated for doubling the number of function evaluations.

The performance of the method is reasonably good, although it is not competitive with a high-order-product Gauss rule using a comparable number of nodes. It might be a useful alternative for use on programmable hand calculators which do not have the memory to store tables of weights and nodes and where the use of only one loop in the algorithm is a significant advantage.

I also plan to investigate the effect of the symmetrization in higher-dimensional calculations, but in such cases the number of nodes increases very rapidly with the dimensionality.

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ON A PROBLEM OF S. J. BEZUSZKA AND M. J. KENNEY ON CYCLIC DIFFERENCE OF PAIRS OF INTEGERS

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Begin with four nonnegative integers, for example, a, b, c, and d. Take cyclic difference of pairs of integers (the smaller integer from the larger), where the fourth difference is always the difference between the last integer

d and the first integer a. Repeat this process on the differences. If we start with 8, 3, 5, 6 and follow the procedure described above, then the process terminates in the sixth row with all zeros. Now we have the following problem due to S. J. Bezuszka and M. J. Kenney [1].

Problem: Is there a selection procedure that will yield sets of four starting integers which terminate with all zeros on the 7th row, the 8th row, ..., the nth row?

Bezuszka and Kenney are of the opinion that the solution to this problem is an interesting application of Tribonacci numbers.

First we note the following easy facts, which we shall use later.

1. If we start with a set of four nonnegative integers—a, b, c, d—that terminates with all zeros on the ith row, then the set of four integers $a + x$, $b + x$, $c + x$, $d + x$, where x is a positive integer, also terminates with all zeros on the ith row.

2. The set of four nonnegative integers a, b, c, d gives the same number of rows as the set na, nb, nc, nd, where n is a positive integer.

3. The set $x - a$, $x - b$, $x - c$, $x - d$ yields the same number of rows as a, b, c, d, provided none of $x - a$, $x - b$, $x - c$, $x - d$ is a negative integer. Again, the set $x - a$, $x - b$, $x - c$, $x - d$ yields the same number of rows as a, b, c, d. We can take the integer x big enough to make each of $x - a$, $x - b$, $x - c$, and $x - d$ nonnegative.

4. If in place of a, b, c, d any cyclic or reverse cyclic order of a, b, c, d is taken as the set of four starting numbers, we again get the same number of rows. For example, 0, 0, a, b being the reverse cycle of 0, 0, b, a will terminate in the same number of steps.

From the above, it is clear that any set of four nonnegative integers a, b, c, d can be replaced by the set 0, u, v, w, which yields the same number of rows as a, b, c, d.

Let a, b, c, d be the four starting numbers. Denote a_1, b_1, c_1, d_1 as r_2; a_2, b_2, c_2, d_2 as r_3; ...; and A_1, B_1, C_1, D_1 as R_2; A_2, B_2, C_2, D_2 as R_3; ... . For example

$$\begin{array}{c} \cdots\cdots\cdots \\ A_1,\ B_1,\ C_1,\ D_1 \\ a,\ b,\ c,\ d \\ a_1,\ b_1,\ c_1,\ d_1 \\ \cdots\cdots\cdots \end{array}$$

Suppose we are given four nonnegative integers a, b, c, d. Is it always possible to find R_2? That is, can we find four nonnegative integers t, u, v, w that will yield a, b, c, d in the second row?

If we start with four nonnegative integers t, u, v, w as our first row, where $t+u+v+w$ is either odd or even, and get a, b, c, d in the second row, then it is easy to see that $a+b+c+d$ is always even. So a, b, c, d with an odd total can never be the second row of any set of four nonnegative integers t, u, v, w. Hence, R_2 is not possible if $a+b+c+d$ is odd. Again, R_3 is not possible if a, b, c, d are such that a and b are odd (even) and c and d are even (odd), for then, if R_2 exists, R_2 will have three odd and one even or one odd and three even, thereby making $A_1 + B_1 + C_1 + D_1$ odd and R_3 impossible.

If the four starting numbers are a, b, c, d and R_2 exists for this set of numbers, then after a little calculation it can be seen that we must have one of the following situations:

(i) $a = b + c + d$
(ii) $b = a + c + d$
(iii) $c = a + b + d$
(iv) $d = a + b + c$
(v) $a + b = c + d$
(vi) $a + c = b + d$
(vii) $a + d = b + c$

Hence, if we are given a, b, c, d where none of the above seven cases holds, then R_2 is impossible.

Since any set of four nonnegative integers t, u, v, w can be replaced by 0, a, b, c ($c \geq a$) without changing the number of steps, from now on, we take 0, a, b, c ($c \geq a$) as our starting numbers.

In case the four starting numbers are 0, a, b, c ($c \geq a$), then R_2 is possible if either $b = c + a$ or $c = a + b$. If we have 0, a, $a+c$, c, we can take R_2 as

(i) $a, a, 0, a + c$ (iii) $a + c, a + c, c, a + 2c$ or
(ii) $c, c, a + c, 0$ (iv) $a + c, a + c, 2a + c, a$

If we have 0, a, b, $a+b$, we can take R_2 as

(i) $0, 0, a, a + b$ (iii) $a, a, 2a, 2a + b$ or
(ii) $a + b, a + b, b, 0$ (iv) $b, b, a + b, a + 2b$

The two sets of four starting numbers a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2 are said to be complements of each other if $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = d_1 + d_2$. If two sets of four starting numbers are complements of each other, they terminate on the same number of rows. Now $a, a, 0, a + c$ and $c, c, a + c, 0$ are complements of each other and $0, 0, a, a + b$ and $a + b, a + b, b, 0$ are complements of each other.

Theorem 1: If the set of four nonnegative integers 0, a, b, c, where $c \geq a + b$ terminates in k steps, then the set of four integers 0, $c - b$, $2c - b$, $4c - b - a$ terminates in $k + 3$ steps.

Proof: Let the four starting numbers be 0, $c - b$, $2c - b$, $4c - b - a$. They are clearly nonnegative. Then we have

$$\begin{array}{cccc} 0, & c-b, & 2c-b, & 4c-b-a \\ c-b, & c, & 2c-b, & 4c-b-a \\ b, & c-a, & 2c-b, & 3c-a \\ c-a-b, & c+a-b, & c+b-a, & 3c-a-b \end{array}$$

The fourth row can be rewritten as x, $2a+x$, $2b+x$, $2c+x$ where $x = c - a - b$, a nonnegative integer. Now, the four starting integers x, $2a+x$, $2b+x$, $2c+x$ will take the same number of steps as 0, $2a$, $2b$, $2c$ for termination. Again, 0, $2a$, $2b$, $2c$ will yield the same number of steps as 0, a, b, c. Thus the set 0, $c - b$, $2c - b$, $4c - b - a$ needs three steps more than 0, a, b, c for termination. Hence, the theorem is proved.

Since $4c - b - a \geq (c - b) + (2c - b) - 3c - 2b$, taking 0, $c - b$, $2c - b$, $4c - b - a$ as 0, a_1, b_1, c_1, where $c_1 \geq a_1 + b_1$, we can get four nonnegative integers 0, $c_1 - b_1$, $2c_1 - b_1$, and $4c_1 - b_1 - a_1$ which will yield three steps more than 0, $c - b$, $2c - b$, $4c - b - a$. We can continue this process n times to get $3n$ steps more than the number of steps given by 0, a, b, c.

If we have 0, a, b, c, where $c < a + b$ but greater than each of a and b, then we consider the reverse cycle of its complement c, $c - a$, $c - b$, 0, that is, 0, $c - b$, $c - a$, c. Now Theorem 1 can be applied to 0, $c - b$, $c - a$, c for $c > (c - b) + (c - a)$.

Theorem 2: If the set of four nonnegative integers 0, a, 0, b, where $b > a$, terminates in k steps, then the set of four integers 0, $a + b$, $a + 2b$, $a + 4b$ terminates in $k + 3$ steps. If $a > b$, we can take 0, $2b$, $3b$, $a + 4b$.

Proof: The proof is easy and is left to the reader.

Since $a + 4b > (a + b) + (a + 2b)$ for $b > a$, we can apply Theorem 1 to the new set. Hence, if we start with 0, a, 0, b, $b > a$, which terminates on the 5th row, we get two different sets of four starting numbers, one from Theorem 1 and the other from Theorem 2, each of which terminates on the 8th row. They are given by $0, b, 2b, 4b - a$ and $0, a + b, a + 2b, a + 4b$. Their reversed complements, given by $0, 2b - a, 3b - a, 4b - a$ and $0, 2b, 3b, 4b + a$ will also terminate on the 8th row.

Since $0, a, 0, b$ and $0, a + x, x, b + 2x$ $(b > a)$ have the same number of steps, we get another set $0, b + x, 2b + 3x, 4b - a + 6x$, by Theorem 1, which also terminates on the 8th row. Again, $0, b, 2b, 4b - a$ and $0, b + x, 2b + x, 4b - a + 2x$ have the same number of steps.

We give examples of some sets of four integers that terminate on the 3rd, 4th, 5th, 6th, and 7th row. We have not included their complements in our list.

1. $0, 0, 0, a$ $(a > 0)$ .... five rows

2. $0, 0, a, a$ $(a > 0)$ .... four rows
   $0, 0, a, a + b$; $0, 0, a + x, a + b + 2x$ $(0 < b \leq a)$ ...... five rows
   $0, 0, a, 2a + x$ $(x \quad 0)$; $0, 0, a, na + x$ $(n \geq 3)$ .... seven rows

3. $0, a, 0, a$ $(a > 0)$; $0, a, 2a, a$ $(a > 0)$ .... three rows
   $0, a, a + x, x$ $(x \neq a)$; $0, a + x, x, a + 2x$ $(x > 0)$;
   $0, a + x, a, 2a + x$ $(a > 0)$ .... four rows
   $0, a, a, 2a + x$ $(x > 0)$; $0, a, a, 2a - x$ $(x \leq a)$ .... five rows
   $0, a, 0, b$ $(a \neq b$, not both zero);
   $0, a + x, x, b + 2x$ $(b \neq a$, not both zero)
   $0, a + x, a, a + 2x$ .... five rows

4. $0, a, a + x, 2a + x$; $0, x, a + x, a + 2x$ $(a,x > 0)$ .... six rows
   $0, a, 2a, 5a$ $(a > 0)$; $0, 3a, 4a, 5a$ $(a > 0)$ .... six rows
   $0, a + x, 2a + x, 3a + x$ $(a \neq 0, x > 2a)$;
   $0, a - x, 2a - x, 3a - x$ $(a \leq x < a)$ .... six rows
   $0, a, a + x, a + x$ $(x \geq a > 0)$ .... five rows
   $0, a, a + x, a + x$ $(x < a)$; $0, 3a, 5a, 4a$ $(a > 0)$ .... seven rows

The above list contains many sets of four nonnegative integers $0, a, b, c$ where $c \geq a + b$. Hence, Theorem 1 can be applied to any of these sets to get three rows more than the particular set of four numbers has. For example,

(i) $0, a, 0, a \rightarrow 0, a, 2a, 3a \rightarrow 0, a, 4a, 9a \rightarrow \ldots$ can be continued $n - 1$ times to get $3n$ steps.

(ii) $0, 0, a, a$ $(a > 0) \rightarrow 0, 0, a, 3a \rightarrow 0, 2a, 5a, 11a \rightarrow \ldots$ can be continued $n - 1$ times to get $3n + 1$ steps.

(iii) $0, 0, 0, a$ $(a > 0) \rightarrow 0, a, 2a, 4a \rightarrow 0, 2a, 6a, 13a \rightarrow \ldots$ can be continued $n - 1$ times to get $3n + 2$ steps.

Hence, we have a selection procedure that will yield sets of four starting numbers that will terminate with all zeros on the nth row, $n = 6, 7, 8, \ldots$.

Below we note some interesting facts:

1. $0, a, a, a$ $(a > 0)$; $0, a, a, a + x$ $(x \neq a)$; $0, a, a + x, a$ $(x \neq a)$; and $0, a, a + x, a + x$ $(x \geq a > 0)$ have five rows.

2. $0, b, 2b, 4b - a$ $(b > 0)$ and $0, b + x, 2b + x, 4b + 2x$ have eight steps.

3. $0, x, a + x, a + b + 2x$ gives three steps more than $0, 0, a, a + b$ $(0 < b \leq a, x > a)$.
   $0, 1, 1 + a, 1 + a + a^2$ $(a > 2)$ gives three steps more than $1, 0, a, a + 1$.

We know that $0, s, 0, s$ $(s \neq 0)$ terminates on the 3rd row. We can write $s = b - a$ in many ways. Then $0, b - a, 2b + 2x, 3b \pm a + 4x$ gives three steps more than $0, b - a, 0, b - a$. Similarly, $0, m, 2m - \ell, 5m - 3\ell + x$ gives three steps more than $s, 0, s, 2s$, where $s = m - \ell$. Again, $0, a, 2a, 5a$ and $0, 3a,$

$4a$, $5a$ gives three steps more than a, 0, a, $2a$. Hence, we can have many sets of four numbers of the form 0, a, b, c having the same number of steps.

However, we can tell the number of steps of the reduced set 0, a, b, c in the following cases:

0, 0, 0, a ($a > 0$) five rows; 0, 0, a, a ($a > 0$) four rows;
0, 0, a, b ($a < b \leq 2a$) five rows; 0, 0, a, $2a + x$ ($x > 0$) seven rows;
0, 0, a, $na + x$ ($n \geq 3$) seven rows; 0, a, 0, a ($a > 0$) three rows;
0, a, 0, b ($a \neq b$) five rows; 0, a, b, c ($b = a + c$, $a = c > 0$) three rows;
0, a, b, c ($b = a + c$, $a \neq c$) four rows;
0, a, b, c ($c = a + b$, $a = b > 0$) four rows;
0, a, b, c ($c = a + b$, $a < b$) six rows; and
0, a, b, c ($c = a + b$, $a > b$) four rows.

From the above, it is clear that the only case which presents difficulty in deciding the number of steps without actual calculation is

$$0,\ a,\ b,\ c\ (abc \neq 0,\ b \neq a + c,\ c \neq a + b),$$

where we can assume $a < c$.

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ASYMPTOTIC BEHAVIOR OF LINEAR RECURRENCES

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In general, it is difficult to predict at a glance the ultimate behavior of a linear recurrence sequence. For example, in some problems where the sequence represents the value of a physical quantity at various times, we might want to know if the sequence is always positive, or at least positive from some point on.

Consider the two sequences:

$$w_0 = 3,\ w_1 = 3.01,\ w_2 = 3.0201$$

and

$$w_{n+3} = 3.01w_{n+2} - 3.02w_{n+1} + 1.01w_n \qquad \text{for } n \geq 0;$$

$$v_0 = 3,\ v_1 = 3.01,\ v_2 = 3.0201$$

and

$$v_{n+3} = 3v_{n+2} - 3.01v_{n+1} + 1.01v_n \qquad \text{for } n \geq 0.$$

The sequence $\{w_n\}$ is always positive, but the sequence $\{v_n\}$ is infinitely often positive and infinitely often negative. This last fact is not obvious from looking at the first few terms of $\{v_n\}$ since the first negative term is v_{735}.