BINARY WORDS WITH MINIMAL AUTOCORRELATION AT OFFSET ONE

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In various communications problems, it has been found to be advantageous to make use of binary words with the property that at offset one they have aucorrelation value as small in magnitude as possible. The purpose of this paper is to derive the means and variances for the autocorrelations of these words at all possible offsets. The derivations are combinatorial in nature, and several new combinatorial identities are obtained.

1. AUTOCORRELATIONS AT VARIOUS OFFSETS

We consider two methods of autocorrelating binary words. The *cyclic* autocorrelation at offset d is defined to be the number of agreements minus the number of disagreements between the original word and a cyclic shift of itself by d places. If the word is $\underline{v} = (v_0, v_1, \ldots, v_{L-1})$, and if subscripts are reduced modulo L, then this autocorrelation is given by

$$\tau_d(\underline{v}) = \sum_{i=0}^{L-1} (-1)^{v_i + v_{i+d}}. \tag{1.1}$$

The *truncated* autocorrelation at offset d is the number of agreements minus the number of disagreements between the last $L-d$ bits of the original word and a right cyclic shift of the word by d places. The formula for this autocorrelation is

$$\tau_d^*(\underline{v}) = \sum_{i=0}^{L-d-1} (-1)^{v_i + v_{i+d}}. \tag{1.2}$$

Note that $\tau_d(\underline{v}) = \tau_d^*(\underline{v}) + \tau_{L-d}^*(\underline{v})$. By symmetry, $\tau_d(\underline{v}) = \tau_{L-d}(\underline{v})$. Thus, for $d > L/2$, we can compute $E(\tau_d)$, $E(\tau_d^2)$, and $E(\tau_d^*)$ from their values with $d \le L/2$. $E(\tau_d^{*2})$ needs special treatment. Therefore, unless stated otherwise, we assume $d \le L/2$.

Our principal result is

Theorem 1.1: Let L be a positive integer, and let $\underline{v}$ range over the binary L-tuples with minimal cyclic autocorrelation at offset 1. Then, for $1 \le d \le [L/2]$,

$$E(\tau_d) = \begin{cases} L\left(\dfrac{1+(-1)^d}{2}\right)(-1)^{d/2}\dbinom{L/2}{d/2}\Big/\dbinom{L}{d}, & L \equiv 0 \pmod 4 \\ L(-1)^{[d/2]}\dbinom{(L-1)/2}{[d/2]}\Big/\dbinom{L}{d}, & L \equiv 1 \pmod 4 \\ (L-2d)\left(\dfrac{1+(-1)^d}{2}\right)(-1)^{d/2}\dbinom{L/2}{d/2}\Big/\dbinom{L}{d}, & L \equiv 2 \pmod 4 \\ L(-1)^{[(d+1)/2]}\dbinom{(L-1)/2}{[d/2]}\Big/\dbinom{L}{d}, & L \equiv 3 \pmod 4 \end{cases}$$

If $L \equiv 2 \pmod 4$ and d is odd, and if we restrict $\underline{v}$ to range over those binary L-tuples satisfying $\tau_1(\underline{v}) = 2c$, $c = \pm 1$, then

$$E(\tau_d) = 2c(d+1)(-1)^{(d-1)/2}\binom{L/2}{(d+1)/2}\Big/\binom{L}{d}.$$

Also,

$$E(\tau^2) = \begin{cases} \dfrac{L^2}{L+2}\left[(-1)^d(L-2d+1)\dbinom{L/2}{d}\Big/\dbinom{L}{2d} + 1\right], & \text{if } L \equiv 0 \pmod 4; \\[2ex] \dfrac{1}{L+3}\left[\dfrac{(-1)^d(L-2d)(L^2-2dL+2L-2d-1)\dbinom{(L-1)/2}{d}}{\dbinom{L-1}{2d}} + L(L+1)\right], & \text{if } L \text{ is odd}; \\[2ex] \dfrac{1}{L+4}\left[(-1)^d(L-2d+1)(L^2-4dL+2L-8d-4)\dbinom{L/2}{d}\Big/\dbinom{L}{2d} + L^2 + 2L + 4\right], & \text{if } L \equiv 2 \pmod 4). \end{cases}$$

If $\underline{v}$ ranges over the binary L-tuples with minimal truncated autocorrelations at offset 1, then, for $1 \le d \le [L/2]$,

$$E(\tau_d^*) = (-1)^{d/2}\left(\frac{1+(-1)^d}{2}\right)(L-d)\binom{[(L-1)/2]}{d/2}\Big/\binom{L-1}{d}.$$

If L is even and $\underline{v}$ is constrained to range over the binary L-tuples satisfying $\tau_1^*(\underline{v}) = c$, $c = \pm 1$, then

$$E(\tau_d^*) = c^d(-1)^{[d/2]}(L-d)\binom{(L-2)/2}{[d/2]}\Big/\binom{L-1}{d}.$$

Also,

$$E(\tau_d^{*2}) = \begin{cases} \dfrac{1}{(L+2)(L+4)}\Big[(-1)^d(L-2d+1)(L^3-2dL^2+4L^2-8dL-4d-2) \\ \qquad \dbinom{L/2}{d}\Big/\dbinom{L}{2d} + L^3 - dL^2 + 4L^2 - 4dL + 2L + 2\Big], & \text{if } L \text{ is even}; \\[2ex] \dfrac{1}{(L+1)(L+3)}\Big[(-1)^d L(L^3-2dL^2+3L^2-4dL+L+2d+1) \\ \qquad \dbinom{(L+1)/2}{d}\Big/\dbinom{L+1}{2d} + L^3 - dL^2 + 2L^2 - 2dL - L + 3d\Big], & \text{if } L \text{ is odd}. \end{cases}$$

If $d > L/2$,

$$E(\tau_d^{*2}) = \begin{cases} \dfrac{1}{(L+2)(L+4)}\Big[(-1)^d 2(L+1)(-2L+2d-1)\dbinom{(L+2)/2}{L-d}\Big/\dbinom{L+2}{2(L-d)} \\ \qquad + L^3 - dL^2 + 4L^2 - 4dL + 2L + 2\Big], & \text{if } L \text{ is even}; \\[2ex] \dfrac{1}{(L+1)(L+3)}\Big[(-1)^{d+1}2L(-2L+2d-1)\dbinom{(L+1)/2}{L-d}\Big/\dbinom{L+1}{2(L-d)} \\ \qquad + L^3 - dL^2 + 2L^2 - 2dL - L + 3d\Big], & \text{if } L \text{ is odd}. \end{cases}$$

2. MINIMIZING AUTOCORRELATION AT OFFSET ONE

Suppose $\underline{v} = (v_0, v_1, \ldots, v_{L-1})$. If we change one bit, say v_i, to obtain $\hat{\underline{v}} = (v_0, \ldots, 1 - v_i, \ldots, v_{L-1})$, then $\tau_d(\hat{\underline{v}}) = \tau_d(\underline{v})$ or $\tau_d(\underline{v}) \pm 4$, because the sign of the two terms, $(-1)^{v_i + v_{i+d}}$ and $(-1)^{v_{i-d} + v_i}$, in the sum $\tau_d(\underline{v})$ have been changed. Since any binary L-tuple may be obtained from any other by changing

$k \leq L$ bits, one at a time, it follows that

$$\tau_d(\underline{v}) \equiv L \pmod 4.$$

A similar argument shows

$$\tau_d^*(\underline{v}) \equiv L - d \pmod 2.$$

Now, if L is odd, the sum (1.1) contains an odd number of terms; thus, $|\tau_d(\underline{v})| \geq 1$. In particular, $|\tau_1(\underline{v})| \geq 1$. The sum (1.2) contains an even number of terms if $d = 1$, so $\tau_1^*(\underline{v})$ may be 0. If $L \equiv 2 \pmod 4$, then $|\tau_1(\underline{v})| \geq 2$, $|\tau_1^*(\underline{v})| \geq 1$. If $L \equiv 0 \pmod 4$, $\tau_1(\underline{v})$ can be 0, while $|\tau_1^*(\underline{v})| \geq 1$.

Let $a_i = v_i \oplus v_{i+1}$ ("$\oplus$" denotes addition modulo 2), $0 \leq i \leq L - 1$, and let $\underline{a} = (a_0, a_1, \ldots, a_{L-1})$. It follows that

$$w(a) = \sum_{i=0}^{L-1} a_i \equiv 0 \pmod 2,$$

so that a_{L-1} is not independent of $a_0, a_1, \ldots, a_{L-2}$. Also, given v_r and a_0, $a_1, \ldots, a_{L-2}$, the vector $\boldsymbol{v}$ is completely determined by the relation

$$v_j = v_r + \sum_{i=\min(j,r)}^{\max(j,r)-1} a_i \pmod 2.$$

In particular,

$$v_i + v_{i+d} \equiv \sum_{k=1}^{d} a_{i+k-1} \pmod 2,$$

so that

$$\tau_d(\underline{v}) = \sum_{i=0}^{L-1} (-1)^{\sum_{k=1}^{d} a_{i+k-1}},$$

$$\tau_d^*(\underline{v}) = \sum_{i=0}^{L-d-1} (-1)^{\sum_{k=1}^{d} a_{i+k-1}}.$$

The case $d = 1$ reduces to

$$\tau_1(\underline{v}) = c = \sum_{i=0}^{L-1} (-1)^{a_i} = L - 2\sum_{i=0}^{L-1} a_i,$$

$$\sum_{i=0}^{L-1} a_i = (L - c)/2;$$

that is, $\underline{a}$ has density $(L - c)/2$. This allows us to count the number $N(c)$ of vectors $\boldsymbol{v}$ with $\tau_1(\underline{v}) = c$:

$$N(c) = \begin{cases} 2\binom{L}{(L-c)/2} & \text{if } L \equiv c \pmod 4; \\ 0 & \text{otherwise.} \end{cases}$$

The factor 2 before the binomial coefficient $\binom{L}{(L-c)/2}$ appears because both $\underline{v}$ and $\overline{\underline{v}}$ (the mod 2 complement of $\underline{v}$) give rise to the same vector $\underline{a}$. Likewise,

$$\tau_1^*(\underline{v}) = c = \sum_{i=0}^{L-2} (-1)^{a_i} = L - 1 - 2\sum_{i=0}^{L-2} a_i,$$

$$\sum_{i=0}^{L-2} a_i = (L - c - 1)/2.$$

Now, $\underline{a}$ has density $w(\underline{a}) = (L - c - 1)/2 + a_{L-1}$, and, since $w(\underline{a})$ is even,

$$w(\underline{a}) = (L - c - 1)/2 + a_{L-1} = \begin{cases} (L - c + 1)/2 & \text{if } (L - c - 1)/2 \text{ is odd;} \\ (L - c - 1)/2 & \text{if } (L - c - 1)/2 \text{ is even;} \end{cases}$$

$$= 2\left[\frac{\frac{L - c - 1}{2} + 1}{2}\right] = 2\left[\frac{L - c + 1}{4}\right].$$

Therefore, the number $N^*(c)$ of vectors $\underline{v}$ satisfying $\tau_1^*(\underline{v}) = c$ is given by

$$N^*(c) = \begin{cases} 2\binom{L}{2[(L - c + 1)/4]} & \text{if } L \equiv c + 1 \pmod 2 \\ 0 & \text{otherwise.} \end{cases}$$

3. THE DISTRIBUTION OF THE CYCLIC AUTOCORRELATIONS τ_d

We now derive the quantities $E(\tau_d(\underline{v}))$ (E: = expected value) and $E(\tau_d^2(\underline{v}))$ when $\underline{v}$ is restricted to the set

$$S(c) = \{\underline{v}: \ \tau_1(\underline{v}) = c\}.$$

Various identities used in the derivation may be found in the Appendix with their proofs. We assume throughout that the binary vectors $\underline{v}$ have length L, and that $L \equiv c \pmod 4$. Of special interest, of course, are the cases $|c| \le 2$, corresponding to vectors with minimal autocorrelation at offset 1. Therefore, we assume that $|c|$ is minimal.

We have shown that, in studying the quantities $\tau_d(\underline{v})$ with $|\tau_1(\underline{v})|$ least possible, we may restrict our attention to the set of vectors

$$R = \left\{\underline{a} = (a_0, \ldots, a_{L-1}): \ \sum_{i=0}^{L-1} a_i = (L - c)/2\right\},$$

where

$$c = \begin{cases} 0 & \text{if } L \equiv 0 \pmod 4 \\ 1 & \text{if } L \equiv 1 \pmod 4 \\ \pm 2 & \text{if } L \equiv 2 \pmod 4 \\ -1 & \text{if } L \equiv 3 \pmod 4. \end{cases}$$

Note that $|R| = \binom{L}{(L - c)/2}$. Let

$$U(d, L) = \sum_{a \in R} \sum_{j=0}^{L-1} (-1)^{\sum_{k=1}^{d} a_{j+k-1}} = \sum_{j=0}^{L-1} \sum_{r=0}^{d} (-1)^r \binom{d}{r}\binom{L - d}{(L - c)/2 - r}$$

$$= L \sum_{r=0}^{d} (-1)^r \binom{d}{r}\binom{L - d}{(L - c)/2 - r},$$

since, for any j, $0 \le j \le L - 1$, $\binom{d}{r}\binom{L - d}{(L - c)/2 - r}$ is the number of d-tuples $(a_j, \ldots, a_{j+d-1})$ of density r. To obtain $E(\tau_d)$, we must divide $U(d, L)$ by $|R|$. We now proceed to determine the quantities $U(d, L)$ for $1 \le d \le L/2$.

Case 1: $c = 0$, $L \equiv 0 \pmod 4$

We make use of Identity 1 (Appendix) to write

$$U(d, L) = L(-1)^{d/2}\left(\frac{1 + (-1)^d}{2}\right)\binom{L/2}{d/2}\binom{L}{L/2}\Big/\binom{L}{d}.$$

Case 2: $c = 1$, $L \equiv 1 \pmod 4$

Identity 2 (Appendix) gives us

$$U(d, L) = L(-1)^{[d/2]}\binom{(L-1)/2}{[d/2]}\binom{L}{(L-1)/2}\Big/\binom{L}{d}.$$

Case 3: $c = -1$, $L \equiv 3 \pmod 4$

We reverse the order of summation to obtain

$$\sum_{r=0}^{d}(-1)^r\binom{d}{r}\binom{L-d}{(L+1)/2-r} = \sum_{s=0}^{d}(-1)^{d-s}\binom{d}{s}\binom{L-d}{(L-1)/2-s},$$

the same sum we considered in Case 2, except for the factor $(-1)^d$. Therefore,

$$U(d, L) = L(-1)^{[(d+1)/2]}\binom{(L-1)}{[d/2]}\binom{L}{(L-1)/2}\Big/\binom{L}{d}.$$

Case 4: $c = 2$, $L \equiv 2 \pmod 4$

Identity 3 (Appendix) yields

$$U(d, L) = \frac{(-1)^{[d/2]}\binom{L}{L/2}\binom{L/2}{[d/2]}}{\binom{L}{2[d/2]}} \cdot \frac{[(L/2)(1+(-1)^d) - 2d(-1)^d]}{L+1}.$$

Case 5: $c = -2$, $L \equiv 2 \pmod 4$

Again, reversing the order of summation in the sum of Case 4 yields

$$U(d, L) = \frac{(-1)^{[(d+1)/2]}\binom{L}{L/2}\binom{L/2}{[d/2]}}{\binom{L}{2[d/2]}} \quad \frac{[(L/2)(1+(-1)^d) - 2d(-1)^d]}{L+1}.$$

Combining the results of Cases 1-5 gives $E(\tau_d)$ in Theorem 2.1. We now proceed with the computation of $E(\tau_d^2)$. Let

$$S(d, L) = \sum_{\underline{a}\in R}\tau_d^2(\underline{v}).$$

Then

$$S(d, L) = \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}}\sum_{i=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{i+k}}$$

$$= \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}\sum_{i=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+k}}$$

$$= \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}\sum_{i=0}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}}$$

$$= \sum_{\underline{a}\in R}\sum_{j=0}^{L-1}\left\{\sum_{i=0}^{d-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}} + \sum_{i=d}^{L-d}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}} + \sum_{i=L-d+1}^{L-1}(-1)^{\sum_{k=0}^{d-1}a_{j+k}+a_{i+j+k}}\right\}.$$

This decomposition of the innermost summation is made to facilitate the summing of the expressions involved. We change the order of summation in each of the three triple sums so that, in each case, the innermost sum is over all $\underline{a} \in R$. Then, for each i and j, we group together all those vectors $\underline{a}$ satisfying

$$r = \sum_{k=0}^{d-1} (a_{j+k} \oplus a_{i+j+k}).$$

If $0 \le i \le d - 1$, then the number of vectors $\underline{a} \in R$ satisfying this condition is

$$\binom{2i}{r}\binom{L-2i}{(L-c)/2-r} = \binom{L}{(L-c)/2}\binom{(L-c)/2}{r}\binom{(L+c)/2}{2i-r}\Big/\binom{L}{2i}.$$

Analogous results hold when $d \le i \le L - d$ and $L - d + 1 \le i \le L - 1$. Thus, we have

$$\begin{aligned}S(d, L) = \sum_{j=0}^{L-1}\Bigg\{&\sum_{i=0}^{d-1}\sum_{r=0}^{2i}(-1)^r\binom{L}{(L-c)/2}\binom{(L-c)/2}{r}\binom{(L+c)/2}{2i-r}\Big/\binom{L}{2i}\\ &+\sum_{i=d}^{L-d}\sum_{r=0}^{2d}(-1)^r\binom{L}{(L-c)/2}\binom{(L-c)/2}{r}\binom{(L+c)/2}{2d-r}\Big/\binom{L}{2d}\\ &+\sum_{i=L-d+1}^{L-1}\sum_{r=0}^{2L-2i}(-1)^r\binom{L}{(L-c)/2}\binom{(L-c)/2}{r}\\ &\qquad\cdot\binom{(L+c)/2}{2L-2i-r}\Big/\binom{L}{2L-2i}\Bigg\}.\end{aligned}$$

The summand is independent of j, so

$$\begin{aligned}S(d, L) = L\binom{L}{(L-c)/2}\Bigg\{&\sum_{i=0}^{d-1}\left[\sum_{r=0}^{2i}(-1)^r\binom{(L-c)/2}{r}\binom{(L+c)/2}{2i-r}\right]\Big/\binom{L}{2i}\\ &+(L-2d+1)\left[\sum_{r=0}^{2d}(-1)^r\binom{(L-c)/2}{r}\binom{(L+c)/2}{2d-r}\right]\Big/\binom{L}{2d}\\ &+\sum_{i=L-d+1}^{L-1}\left[\sum_{r=0}^{2L-2i}(-1)^r\binom{(L-c)/2}{r}\binom{(L+c)/2}{2L-2i-r}\right]\Big/\binom{L}{2L-2i}\Bigg\}.\end{aligned}$$

As above, we divide our calculation into five cases.

<u>*Case 1:*</u> $c = 0$, $L \equiv 0 \pmod 4$

Applying Identity 6 (Appendix), we obtain

$$\begin{aligned}S(d, L) = L\binom{L}{L/2}\Bigg\{&\sum_{i=0}^{d-1}(-1)^i\binom{L/2}{i}\Big/\binom{L}{2i} + (L-2d+1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d}\\ &+\sum_{i=L-d+1}^{L-1}(-1)^{L-i}\binom{L/2}{L-1}\Big/\binom{L}{2L-2i}\Bigg\}\end{aligned}$$

(continued)

$$= L\binom{L}{L/2}\left\{\frac{L+1}{L+2}\left[(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d}+1\right]\right.$$

$$+ (L - 2d + 1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d}$$

$$\left.+\frac{L+1}{L+2}\left[(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} - 1/(L+1)\right]\right\}$$

by Identity 4 (Appendix). Thus,

$$S(d, L) = L^2\binom{L}{L/2}\left\{(-1)^d(L - 2d + 1)\binom{L/2}{d}\Big/\binom{L}{2d} + 1\right\}/(L+2).$$

Case 2: $c = 1$, $L \equiv 1 \pmod 4$

Applying Identity 7 (Appendix), we obtain

$$S(d, L) = L\binom{L}{(L-1)/2}\left\{\sum_{i=0}^{d-1}(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L}{2i} + (L - 2d + 1)(-1)^d\right.$$

$$\left.\binom{(L-1)/2}{d}\Big/\binom{L}{2d} + \sum_{i=L-d+1}^{L-1}(-1)^{L-i}\binom{(L-1)/2}{L-i}\Big/\binom{L}{2L-2i}\right\}$$

$$= L\binom{L}{(L-1)/2}\left\{\frac{L+2}{L+3}\left[(-1)^{d-1}\binom{(L+3)/2}{d}\Big/\binom{L+3}{2d}+1\right]\right.$$

$$+ (L - 2d + 1)(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L}{2d}$$

$$\left.+\frac{L+2}{L+3}\left[(-1)^{d-1}\binom{(L+3)/2}{d}\Big/\binom{L+3}{2d} - 1/(L+2)\right]\right\}$$

by Identity 5 (Appendix). Hence,

$$S(d, L) = \binom{L}{(L-1)/2}\left\{(-1)^d(L - 2d)[L^2 + (-2d+2)L - 2d - 1]\binom{(L-1)/2}{d}\Big/\binom{L-1}{2d}\right.$$

$$\left. + L(L+1)\right\}/(L+3).$$

Case 3: $c = -1$, $L \equiv 3 \pmod 4$

Reversing the order of summation on r in all three sums reduces this to the previous case.

Case 4: $c = 2$, $L \equiv 2 \pmod 4$

Apply Identity 8 (Appendix) to obtain

$$S(d, L) = L\binom{L}{L/2-1}\left\{\sum_{i=0}^{d-1}(-1)^i\binom{L/2}{i}\Big/\binom{L}{2i} - 2\sum_{i=0}^{d-1}(-1)^i\binom{L/2-1}{i-1}\Big/\binom{L}{2i}\right.$$

$$+ (L - 2d + 1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d} - 2(L - 2d + 1)(-1)^d\binom{L/2-1}{d-1}\Big/\binom{L}{2d}$$

$$\left.+ \sum_{i=L-d+1}^{L-1}(-1)^{L-1}\binom{L/2}{L-i}\Big/\binom{L}{2L-2i} - 2\sum_{i=L-d+1}^{L-1}(-1)^{L-1}\binom{L/2-1}{L-i-1}\Big/\binom{L}{2i}\right\}$$

(continued)

$$= L\binom{L}{L/2 - 1}\left\{\frac{L+1}{L+2}(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} + 1 - 2\left(\frac{2(L+1)}{L(L+2)(L+4)}\right)\right.$$

$$\cdot\left[(-1)^{d-1}\{(L+2)d - 1\}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} - 1\right]$$

$$+ (L - 2d + 1)(-1)^d\binom{L/2}{d}\Big/\binom{L}{2d} - 2(L - 2d + 1)(-1)^d\binom{L/2-1}{d-1}\Big/\binom{L}{2d}$$

$$+ \frac{L+1}{L+2}\left[(-1)^{d-1}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} - 1/(L+1)\right] - 2\left(\frac{2(L+1)}{L(L+2)(L+4)}\right)$$

$$\cdot\left[(-1)^{d-1}\{(L+2)d - 1\}\binom{L/2+1}{d}\Big/\binom{L+2}{2d} - 1\right]$$

using Identities 4 and 9 (Appendix). Therefore,

$$S(d, L) = \binom{L}{L/2 - 1}\left\{(-1)^d(L - 2d + 1)(L^2 + [-4d + 2]L - 8d - 4)\right.$$
$$\left.\cdot\binom{L/2}{d}\Big/\binom{L}{2d} + L^2 + 2L + 4\right\}/(L + 4).$$

Case 5: $c = -2$, $L \equiv 2 \pmod 4$

Reversing the order on summation on r in all three sums reduces this to the previous case.

Combining the results of cases 1-5 gives $E(\tau_d^2)$ in Theorem 1.1.

4. THE DISTRIBUTION OF THE TRUNCATED AUTOCORRELATION τ_d^*

We now derive the quantities $E(\tau_d^*(\underline{v}))$ and $E(\tau_d^{*2}(\underline{v}))$ when $\underline{v}$ is restricted to the set

$$S^*(c) = \{\underline{v}:\ \tau_1^*(\underline{v}) = c\}.$$

Various identities used in this derivation may be found in the Appendix, with their proofs. We again assume that the binary vectors $\underline{v}$ have length L, but now $L \equiv c + 1 \pmod 2$. Of special interest, of course, are the cases $|c| \le 1$, corresponding to vectors with minimal autocorrelation at offset 1. Again, we assume that $|c|$ is minimal.

We have shown that, in studying the quantities $\tau_d^*(\underline{v})$ with $|\tau_1^*(\underline{v})|$ least possible, we may restrict our attention to the set of vectors

$$R^* = \left\{\underline{a} = (a_0, a_1, \ldots, a_{L-1}):\ \sum_{i=0}^{L-2} a_i = (L - c - 1)/2\right\},$$

where

$$c = \begin{cases} 0 & \text{if } L \text{ is odd,} \\ \pm 1 & \text{if } L \text{ is even.}\end{cases}$$

Note that

$$|R^*| = \binom{L - 1}{(L - c - 1)/2}.$$

Let

$$U^*(d, L) = \sum_{\underline{a}\in R^*}\sum_{j=0}^{L-d-1}(-1)^{\sum_{k=0}^{d-1} a_{j+k}}$$

Since there are $\binom{d}{r}\binom{L - d - 1}{(L - c - 1)/2 - r}$ d-tuples $(a_j, \ldots, a_{j+d-1})$ of density r, we have

$$U^*(d, L) = \sum_{j=0}^{L-d-1} \sum_{r=0}^{d} (-1)^r \binom{d}{r} \binom{L-d-1}{(L-d-1)/2 - r}$$
$$= (L - d)\sum_{r=0}^{d} (-1)^r \binom{d}{r} \binom{L-d-1}{(L-c-1)/2 - r}$$
$$= \begin{cases} (L-d)(-1)^{d/2} \dfrac{1+(-1)^d}{2} \dbinom{(L-1)/2}{d/2} \dbinom{L-1}{(L-1)/2} \Big/ \dbinom{L-1}{d} & \text{if } c = 0, \text{ by Identity 1;} \\ (L-d)(-1)^{[d/2]} \dbinom{(L-2)/2}{[(d+1)/2]} \dbinom{L-1}{(L-2)/2} \Big/ \dbinom{L-1}{2[(d+1)/2]} & \text{if } c = 1, \text{ by Identity 2;} \\ (L-d)(-1)^{[(d+1)/2]} \dbinom{(L-2)/2}{[(d+1)/2]} \dbinom{L-1}{(L-2)/2} \Big/ \dbinom{L-1}{2[(d+1)/2]} & \text{if } c = -1, \text{ by Identity 2.} \end{cases}$$

To obtain $E(\tau_d^*)$ of Theorem 1.1, we divide $U^*(d, L)$ by $|R^*|$, and combine the cases $c = \pm 1$, to obtain

$$E(\tau_d^*) = (-1)^{d/2} \frac{1+(-1)^d}{2} (L-d) \binom{[(L-1)/2]}{d/2} \Big/ \binom{L-1}{d}.$$

We now proceed with the computation of $E(\tau_d^{*2})$. Let

$$S^*(d, L) = \sum_{\underline{a} \in R^*} \tau_d^{*2}(\underline{v})$$
$$= \sum_{\underline{a} \in R^*} \sum_{j=0}^{L-d-1} (-1)^{\sum_{k=0}^{d-1} a_{j+k}} \sum_{i=0}^{L-d-1} (-1)^{\sum_{k=0}^{d-1} a_{i+k}}$$
$$= \sum_{\underline{a} \in R^*} \sum_{j=0}^{L-d-1} \sum_{i=0}^{L-d-1} (-1)^{\sum_{k=0}^{d-1} a_{j+k} + a_{i+k}}$$
$$= \sum_{\underline{a} \in R^*} \sum_{j=0}^{L-d-1} \sum_{i=-j}^{L-d-j-1} (-1)^{\sum_{k=0}^{d-1} a_{j+k} + a_{i+j+k}}.$$

We split this sum into three parts, depending on the degree of overlap of the two d-tuples $(a_j, \ldots, a_{j+d-1})$ and $(a_{i+j}, \ldots, a_{i+j+d-1})$: complete, partial, or none.

$$S^*(d, L) = \binom{L-1}{(L-c-1)/2} \left\{ \sum_{i=0}^{0} \sum_{j=0}^{L-d-1} 1 + 2 \sum_{i=1}^{d-1} \sum_{j=0}^{L-d-i-1} \sum_{r=0}^{2i} (-1)^r \right.$$
$$\cdot \binom{(L-c-1)/2}{r} \binom{(L+c-1)/2}{2i-r} \Big/ \binom{L-1}{2i}$$
$$\left. + 2 \sum_{i=d}^{L-d-1} \sum_{j=0}^{L-d-i-1} \sum_{r=0}^{2d} (-1)^r \binom{(L-c-1)/2}{r} \binom{(L+c-1)/2}{2d-r} \Big/ \binom{L-1}{2d} \right\}.$$

We now consider three cases, depending on the value of c:

Case 1: $c = 0$, L odd. Applying Identity 6 (Appendix), we obtain,

$$S^*(d, L) = \binom{L-1}{(L-1)/2}\left\{(L-d) + 2\sum_{i=1}^{d-1}(L-d-i)(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L-1}{2i}\right.$$
$$\left. + 2\binom{L-2d+1}{2}(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L-1}{2d}\right\}$$
$$= \binom{L-1}{(L-1)/2}\left\{L - d + 2(L-d)\sum_{i=1}^{d-1}(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L-1}{2i}\right.$$
$$- (L-1)\sum_{i=1}^{d-1}(-1)^i\binom{(L-3)/2}{i-1}\Big/\binom{L-1}{2i}$$
$$\left. + (L-2d+1)(L-2d)(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L+1}{2d}\right\}$$
$$= \binom{L-1}{(L-1)/2}\left\{L - d + 2(L-d)\frac{L}{L+1}\left[(-1)^{d-1}\binom{(L+1)/2}{d}\Big/\binom{L+1}{2} - 1/L\right]\right.$$
$$- (L-1)\frac{2}{(L-1)(L+1)(L+3)}\left[(-1)^{d-1}(dL+d-1)\right.$$
$$\left.\binom{(L+1)/2}{d}\Big/\binom{L+1}{2d} - 1\right]$$
$$\left. + (L-2d+1)(L-2d)(-1)^d\binom{(L-1)/2}{d}\Big/\binom{L-1}{2d}\right\}$$

by Identities 4 and 9. Therefore,

$$S^*(d, L) = \frac{1}{(L+1)(L+3)}\binom{L-1}{(L-1)/2}\left\{(-1)^d L[L^3 + (3-2d)L^2 + (1-4d)L\right.$$
$$\left. + (1+2d)]\binom{(L+1)/2}{d}\Big/\binom{L+1}{2d} + [L^3 + (2-d)L^2 + (-1-2d)L + (3d)]\right\}.$$

Case 2: $c = 1$, L even. Applying Identity 7 (Appendix), we obtain,

$$S^*(d, L) = \binom{L-1}{(L-2)/2}\left\{L - d + 2\sum_{i=1}^{d-1}(L-d-i)(-1)^i\binom{(L-2)/2}{i}\Big/\binom{L-1}{2i}\right.$$
$$\left. + 2\binom{L-2d+1}{2}(-1)^d\binom{(L-2)/2}{d}\Big/\binom{L-1}{2d}\right\}$$
$$= \binom{L-1}{(L-2)/2}\left\{L - d + 2(L-d)\frac{L+1}{L+2}\left[(-1)^{d-1}\binom{(L+2)/2}{d}\Big/\binom{L-2}{2d}\right.\right.$$
$$\left. - 1/(L+1)\right] - 2\left(\frac{L+1}{(L+2)(L+4)}\right)[(-1)^{d-1}(dL+2d-1)$$
$$\left. - (L+2)/(L+1)] + (L-2d+1)(L-2d)(-1)^d\binom{(L-2)/2}{d}\Big/\binom{L-1}{2d}\right\}$$

sing Identities 5 and 10. Consequently,

$$S^*(d, L) = \frac{1}{(L+2)(L+4)}\binom{L-1}{(L-2)/2}\Big\{(-1)^d(L-2d+1)[L^3+(4-2d)L^2$$

$$+(-8d)L+(-2-4d)]\binom{L/2}{d}\Big/\binom{L}{2d}+[L^3+(4-d)L^2+(2-4d)L+2]\Big\}.$$

Case 3: $c = -1$, L even. Reversing the order of summation on r in both sums reduces this to the previous case.

Combining the results of Cases 1–3 gives $E(\tau_d^{*2})$ for $d \le L/2$ in Theorem 1.1. The case when $d > L/2$ is similarly handled, with the result:

$$S^*(d, L) = \binom{L-1}{(L-c-1)/2}\Big\{\sum_{i=0}^{0}\sum_{j=0}^{L-d-1}1 + 2\sum_{i=1}^{L-d-1}\sum_{j=0}^{L-d-i-1}\sum_{r=0}^{2i}(-1)^r$$

$$\cdot\binom{(L-c-1)/2}{r}\binom{(L+c-1)/2}{2i-r}\Big/\binom{L-1}{2i}\Big\}.$$

Once again we consider three cases, depending on the value of c:

Case 1: $c = 0$, L odd. Applying Identity 6, we obtain

$$S^*(d, L) = \binom{L-1}{(L-1)/2}\Big\{L-d+2\sum_{i=1}^{L-d-1}(L-d-i)(-1)^i\binom{(L-1)/2}{i}\Big/\binom{L-1}{2i}$$

$$= \binom{L-1}{(L-1)/2}\Big\{L-d+2(L-d)\frac{L}{L+1}\Big[(-1)^{L-d-1}$$

$$\cdot\binom{(L+1)/2}{L-d}\Big/\binom{L+1}{2(L-d)} - 1/L\Big]$$

$$-\frac{2L}{(L+1)(L+3)}\Big[(-1)^{L-d-1}\{(L+1)(L-d)-1\}$$

$$\binom{(L+1)/2}{L-d}\Big/\binom{L+1}{2(L-d)} - 1\Big]\Big\}$$

using Identities 5 and 9. This yields

$$S^*(d, L) = \binom{L-1}{(L-1)/2}\Big\{(-1)^{L-d}2L(-2L+2d-1)\binom{(L+1)/2}{L-d}\Big/\binom{L+1}{2(L-d)}$$

$$+L^3+(2-d)L^2+(-1-2d)L+3d\Big\}\Big/(L+1)(L+3).$$

Case 2: $c = 1$, L even. Applying Identity 7, we obtain

$$S^*(d, L) = \binom{L-1}{(L-2)/2}\Big\{L-d+2\sum_{i=1}^{L-d-1}(-1)^i(L-d-i)\binom{(L-2)/2}{i}\Big/\binom{L-1}{2i}\Big\}$$

$$= \binom{L-1}{(L-2)/2}\Big\{L-d+2(L-d)\frac{L+1}{L+2}\Big[(-1)^{L-d-1}\binom{(L+2)/2}{L-d}\Big/\binom{L+2}{2(L-d)}$$

$$-1/(L+1)\Big]-\frac{2(L+1)}{(L+2)(L+4)}\Big[(-1)^{L-d-1}\{(L+2)(L-d)-1\}$$

(continued)

$$\cdot \binom{(L+2)/2}{L-d} \Big/ \binom{L+2}{2(L-d)} - 1\Big]\Big\}$$

using Identities 6 and 10. As a result,

$$S^*(d, L) = \binom{L-1}{(L-2)/2}\Big\{(-1)^{L-d}2(L+1)(-2L+2d-1)\binom{(L+2)/2}{L-d}\Big/\binom{L+2}{2(L-d)}$$

$$+ L^3 + (4-d)L^2 + (2-4d)L + 2\Big\}\Big/(L+2)(L+4).$$

Case 3: $c = -1$, L even. Reversing the order of summation on r reduces this to the previous case.

Combining the results of Cases 1-3 gives $E(\tau_d^{*2})$ for $d > L/2$ in Theorem 1.1.

5. VARIANCES

The variances of τ_d and τ_d^* may be obtained from the above results by noting that the variance σ^2 of any statistic x is given by

$$\sigma^2(x) = E(x^2) - E(x)^2.$$

These numbers are tabulated along with $E(\tau_d)$, etc., in Table 1.

TABLE 1. Expected Values for Selected Values of L

L	d	$E(\tau_d)$	$E(\tau_d^2)$	$E(\tau_d^*)$	$E(\tau_d^{*2})$	σ^2	σ^{*2}
4	1	.00000	.00000	.00000	1.00000	.00000	1.00000
4	2	-1.33333	5.33333	-.66667	1.33333	3.55556	.88889
4	3	.00000	.00000	.00000	1.00000	.00000	1.00000
5	1	1.00000	1.00000	.00000	.00000	.00000	.00000
5	2	-1.00000	5.00000	-1.00000	3.66667	4.00000	2.66667
5	3	-1.00000	5.00000	.00000	1.33333	4.00000	1.33333
5	4	1.00000	1.00000	1.00000	1.00000	.00000	.00000
6	1	.00000	4.00000	.00000	1.00000	4.00000	1.00000
6	2	-.40000	4.00000	-.80000	4.00000	3.84000	3.36000
6	3	.00000	10.40000	.00000	2.60000	10.40000	2.60000
6	4	-.40000	4.00000	.40000	1.60000	3.84000	1.44000
6	5	.00000	4.00000	.00000	1.00000	4.00000	1.00000
7	1	-1.00000	1.00000	.00000	.00000	.00000	.00000
7	2	-1.00000	7.40000	-1.00000	5.80000	6.40000	4.80000
7	3	.60000	4.20000	.00000	1.60000	3.84000	1.60000
7	4	.60000	4.20000	.60000	2.60000	3.84000	2.24000
7	5	-1.00000	7.40000	.00000	1.60000	6.40000	1.60000
7	6	-1.00000	1.00000	-1.00000	1.00000	.00000	.00000
8	1	.00000	.00000	.00000	1.00000	.00000	1.00000
8	2	-1.14286	9.14286	-.85714	6.28571	7.83673	5.55102
8	3	.00000	3.65714	.00000	3.51429	3.65714	3.51429
8	4	.68571	12.80000	.34286	3.20000	12.32980	3.08245
8	5	.00000	3.65714	.00000	2.60000	3.65714	2.60000
8	6	-1.14286	9.14286	-.28571	1.71429	7.83673	1.63265
8	7	.00000	.00000	.00000	1.00000	.00000	1.00000

(continued)

TABLE 1 (continued)

L	d	$E(\tau_d)$	$E(\tau_d^2)$	$E(\tau_d^*)$	$E(\tau_d^{*2})$	σ^2	σ^{*2}
9	1	1.00000	1.00000	.00000	.00000	.00000	.00000
9	2	-1.00000	9.57143	-1.00000	7.85714	8.57143	6.85714
9	3	-.42857	6.14286	.00000	3.54286	5.95918	3.54286
9	4	.42857	9.00000	-.42857	5.80000	8.81633	5.61633
9	5	.42857	9.00000	.00000	3.20000	8.81633	3.20000
9	6	-.42857	6.14286	.42857	2.60000	5.95918	2.41633
9	7	-1.00000	9.57143	.00000	1.71429	8.57143	1.71429
9	8	1.00000	1.00000	1.00000	1.00000	.00000	.00000
16	1	.00000	.00000	.00000	1.00000	.00000	1.00000
16	2	-1.06667	17.06667	-.93333	14.66667	15.92889	13.79556
16	3	.00000	13.12821	.00000	10.96923	13.12821	10.96923
16	4	.24615	14.91841	.18462	11.10676	14.85782	11.07268
16	5	.00000	13.52603	.00000	9.61414	13.52603	9.61414
16	6	-.11189	15.31624	-.06993	9.25128	15.30372	9.24639
16	7	.00000	11.37778	.00000	7.75556	11.37778	7.75556
16	8	.08702	28.44444	.04351	7.11111	28.43687	7.10922
16	9	.00000	11.37778	.00000	6.33333	11.37778	6.33333
16	10	-.11189	15.31624	-.04196	5.42222	15.30372	5.42046
16	11	.00000	13.52603	.00000	4.54188	13.52603	4.54188
16	12	.24615	14.91841	.06154	3.64755	14.85782	3.64377
16	13	.00000	13.12821	.00000	2.76410	13.12821	2.76410
16	14	-1.06667	17.06667	-.13333	1.86667	15.92889	1.84889
16	15	.00000	.00000	.00000	1.00000	.00000	1.00000

APPENDIX

In this section we give identities used in the proof of Theorem 1.1. Some are merely stated, and others, previously unknown to the authors, are proved.

Identity 1:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2x-n}{x-k} = (-1)^{n/2}\,\frac{1+(-1)^n}{2}\binom{x}{n/2}\binom{2x}{x} \Big/ \binom{2x}{n}$$

Identity 2:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2x+1-n}{x-k} = (-1)^{[n/2]}\binom{x}{[n/2]}\binom{2x+1}{x} \Big/ \binom{2x+1}{n}$$

Identity 3:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2x+2-n}{x-k}$$

$$= \frac{(-1)^{[n/2]}\binom{2x+2}{x+1}\binom{x+1}{[n/2]}}{\binom{2x+2}{2[n/2]}} \cdot \frac{[(x+1)(1+(-1)^n) - 2n(-1)^n]}{2(x+2)}$$

Identity 4:

$$\sum_{k=a}^{n} (-1)^k \binom{x}{k} \Big/ \binom{2x}{2k} = \frac{2x+1}{2x+2}\left[(-1)^n\binom{x+1}{n+1} \Big/ \binom{2x+2}{2n+2} + (-1)^a\binom{x+1}{a} \Big/ \binom{2x+2}{2a}\right]$$

Identity 5:

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k}\Big/\binom{2x+1}{2k} = \frac{2x+3}{2x+4}\left[(-1)^n\binom{x+2}{n+1}\Big/\binom{2x+4}{2n+2} + (-1)^a\binom{x+2}{a}\Big/\binom{2x+4}{2a}\right]$$

Identity 6:

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{n-k} = (-1)^{n/2}\,\frac{1+(-1)^n}{2}\binom{x}{n/2}$$

Identity 7:

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x+1}{n-k} = (-1)^{[n/2]}\binom{x}{[n/2]}$$

Identity 8:

$$\sum_{k=0}^{2n}(-1)^k\binom{x}{k}\binom{x+2}{2n-k} = (-1)^n\left[\binom{x}{n} - \binom{x}{n-1}\right] = (-1)^n\left[\binom{x+1}{n} - 2\binom{x}{n-1}\right]$$

Identity 9:

$$\sum_{k=a}^{n}(-1)^k\binom{x-1}{k-1}\Big/\binom{2x}{2k}$$

$$= \frac{2(2x+1)}{(2x)(2x+2)(2x+4)}\left\{(-1)^n[2(x+1)(n+1)-1]\binom{x+1}{n+1}\Big/\binom{2x+2}{2n+2}\right.$$

$$\left. + (-1)^a[2a(x+1)-1]\binom{x+1}{a}\Big/\binom{2x+2}{2a}\right\}$$

Identity 10:

$$\sum_{k=a}^{n}(-1)^k k\binom{x}{k}\Big/\binom{2x+1}{2k}$$

$$= \frac{2x+3}{(2x+4)(2x+6)}\left\{(-1)^n[2(x+2)(n+1)-1]\binom{x+2}{n+1}\Big/\binom{2x+4}{2n+2}\right.$$

$$\left. + (-1)^a[2a(x+2)-1]\binom{x+2}{a}\Big/\binom{2x+4}{2a}\right\}$$

Proof of Identity 1: See [1, 3.58].

Proof of Identity 2: Let

$$f(x,\,n) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x+1-n}{x-k}$$

$$= \sum_{k=0}^{n}(-1)^k\left[\binom{n}{k}\binom{2x-n}{x-k} + \binom{n+1}{k+1}\binom{2x-n}{x-1-k} - \binom{n}{k+1}\binom{2x-n}{x-1-k}\right]$$

$$= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x-n}{x-k} - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}\binom{2x+1-(n+1)}{x-k}$$

$$+ \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2x-n}{x-k}$$

$$= (-1)^{n/2}\,[1+(-1)^n]\binom{x}{n/2}\binom{2x}{x}\Big/\binom{2x}{n} - f(x,\,n+1) \quad \text{by Identity 1.}$$

$$f(x,\ 2m) = \sum_{k=1}^{m} [f(x,\ 2k) - f(x,\ 2k-2)] + f(x,\ 0)$$

$$= -2\binom{2x}{x}\sum_{k=1}^{m}(-1)^{k-1}\binom{x}{k-1} \Big/ \binom{2x}{2k-2} + \binom{2x+1}{x}$$

$$= (-1)^{m}\binom{x}{m}\binom{2x+1}{x} \Big/ \binom{2x+1}{2m} \quad \text{by Identity 4.}$$

Thus

$$f(x,\ 2m-1) = (-1)^{m-1}\binom{x}{m-1}\binom{2x+1}{x} \Big/ \binom{2x+1}{2m-1}$$

$$f(x,\ n) = (-1)^{[n/2]}\binom{x}{[n/2]}\binom{2x+1}{x} \Big/ \binom{2x+1}{n} \qquad \text{Q.E.D.}$$

Proof of Identity 3: Let

$$g(x,\ n) = \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{2x+2-n}{x-k}$$

$$= \sum_{k=0}^{n}(-1)^{k}\left(\binom{n}{k}\binom{2x+1-n}{x-k} + \binom{n+1}{k+1}\binom{2x+1-n}{x-1-k} - \binom{n}{k+1}\binom{2x+1-n}{x-1-k}\right)$$

$$= \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{2x+1-n}{x-k} - \sum_{k=0}^{n+1}(-1)^{k}\binom{n+1}{k}\binom{2x+2-(n+1)}{x-k}$$

$$+ \sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{2x+1-n}{x-k}$$

$$= 2(-1)^{[n/2]}\binom{x}{[n/2]}\binom{2x+1}{x} \Big/ \binom{2x+1}{n} - g(x,\ n+1) \quad \text{by Identity 2.}$$

$$g(x,\ 2m) = \sum_{k=1}^{m}[g(x,\ 2k) - g(x,\ 2k-2)] + g(x,\ 0)$$

$$= -2\binom{2x+1}{x}\left[\sum_{k=1}^{m}(-1)^{k}\binom{x+1}{k} \Big/ \binom{2x+2}{2k} + \sum_{k=0}^{m-1}(-1)^{k}\binom{x}{k} \Big/ \binom{2x+1}{k}\right]$$

$$+ \binom{2x+2}{x}$$

$$= -2\binom{2x+1}{x}\left[\frac{2x+3}{2x+4}\left\{(-1)^{m}\binom{x+2}{m} \Big/ \binom{2x+4}{2m+2} - 1/(2x+3)\right\}\right.$$

$$\left. + \frac{2x+3}{2x+4}\left\{(-1)^{m-1}\binom{x+2}{m} \Big/ \binom{2x+4}{2m} + 1\right\}\right] \quad \text{by Identities 4 and 5.}$$

$$g(x,\ 2m) = (-1)^{m}(x-2m+1)\binom{2x+4}{x+2}\binom{x+2}{m} \Big/ \binom{2x+4}{2m}2(2x-2m+3)$$

Thus

$$g(x,\ 2m-1) = (-1)^{m-1}(2m-1)\binom{2x+4}{x+2}\binom{x+2}{m-1} \Big/ \binom{2x+4}{2m-2}2(2x-2m+5)$$

and

$$g(x, n) = (-1)^{[n/2]}\{[x+1][1+(-1)^n] - 2n(-1)^n\}\binom{2x+2}{x+1}\binom{x+1}{[n/2]} \Big/ \binom{2x+2}{2[n/2]}2(x+2)$$

Q.E.D.

Proof of Identity 4:

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k} \Big/ \binom{2x}{2k} = \left[(-1)^x 2^{2x} \Big/ \binom{2x}{x}\right]\sum_{k=a}^{n}\binom{k-1/2}{x} \qquad \text{by [1, Z.55]}$$

$$= \left[(-1)^x 2^{2x} \Big/ \binom{2x}{x}\right]\left[\binom{n+1/2}{x+1} - \binom{a-1/2}{x+1}\right] \qquad \text{by [1, 1.48]}$$

$$= \left[(-1)^x 2^{2x} \Big/ \binom{2x}{x}\right]\left[(-1)^{x-n}2^{-2x-2}\binom{2x+2}{x+1}\binom{x+1}{n+1} \Big/ \binom{2x+2}{2n+2}\right.$$

$$\left. -(-1)^{x-a+1}2^{-2x-2}\binom{2x+2}{x+1}\binom{x+1}{a} \Big/ \binom{2x+2}{2a}\right] \qquad \text{by [1, Z.55]}$$

$$= \frac{2x+1}{2x+2}\left[(-1)^n\binom{x+1}{n+1} \Big/ \binom{2x+2}{2n+2} + (-1)^a\binom{x+1}{a} \Big/ \binom{2x+2}{2a}\right]$$

Q.E.D.

Proof of Identity 5:

$$\sum_{k=a}^{n}(-1)^k\binom{x}{k} \Big/ \binom{2x+1}{2k} = \sum_{k=a}^{n}(-1)^k\binom{x+1}{k} \Big/ \binom{2x+2}{2k}$$

$$= \frac{2x+3}{2x+4}\left[(-1)^n\binom{x+2}{n+1} \Big/ \binom{2x+4}{2n+2} + (-1)^a\binom{x+2}{a} \Big/ \binom{2x+4}{2a}\right]$$

by Identity 4.

Proof of Identity 6: See [1, 3.32].

Proof of Identity 7: Apply [1, 3.31] with $y = x + 1$.

Proof of Identity 8: Apply [1, 3.31] with $y = x + 2$.

Proof of Identity 9:

$$\sum_{k=a}^{n}(-1)^k\binom{x-1}{k-1} \Big/ \binom{2x}{2k} = \frac{2n+1}{2n}\sum_{k=a}^{n}(-1)^k\binom{n+1}{k+1} \Big/ \binom{2x+2}{2k+2} - \frac{1}{2x}\sum_{k=a}^{n}(-1)^k\binom{x}{k} \Big/ \binom{2x}{2k}$$

$$= -\frac{2x+1}{2x}\cdot\frac{2x+3}{2x+4}\left[(-1)^{n+1}\binom{x+2}{n+2} \Big/ \binom{2x+4}{2n+4}\right.$$

$$\left. + (-1)^{a+1}\binom{x+2}{a+1} \Big/ \binom{2x+4}{2a+2}\right)$$

$$-\frac{1}{2x}\cdot\frac{2x+1}{2x+2}\left[(-1)^n\binom{x+1}{n+1} \Big/ \binom{2x+2}{2n+2}\right.$$

$$\left. + (-1)^a\binom{x+1}{a} \Big/ \binom{2x+2}{2a}\right] \qquad \text{by Identity 4}$$

(continued)

$$= \frac{2(2x+1)}{2x(2x+2)(2x+4)}\left\{(-1)^n[2(n+1)(x+1)-1]\binom{x+1}{n+1}\Big/\binom{2x+2}{2n+2}\right.$$

$$\left. + (-1)^a[2a(x+1)-1]\binom{x+1}{a}\Big/\binom{2x+2}{2a}\right\} \qquad \text{Q.E.D.}$$

Proof of Identity 10:

$$\sum_{k=a}^{n}(-1)^k k\binom{x}{k}\Big/\binom{2x+1}{2k} = (x+1)\sum_{k=a}^{n}(-1)^k\binom{x}{k-1}\Big/\binom{2x+2}{2k}$$

$$= \frac{2x+3}{(2x+4)(2x+6)}\left\{(-1)^n[2(n+1)(x+2)-1]\right.$$

$$\binom{x+2}{n+1}\Big/\binom{2x+4}{2n+2}$$

$$\left. + (-1)^a[2a(x+2)-1]\binom{x+2}{a}\Big/\binom{2x+4}{2a}\right\} \text{ by Identity 9}$$

Q.E.D.

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FIBONACCI CUBATURE

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Korobov [1] developed procedures for integration over an N-dimensional cube which are referred to in the literature [2, 3, 4] as number-theoretical methods or the method of optimal coefficients. These methods involve summation over a lattice of nodes defined by a single index instead of N nested summations. For the two-dimensional case, a particularly simple form involving the Fibonacci numbers is obtained. Designating the Nth Fibonacci number by F_N, k/F_N by x_k, and $\{F_{N-1}x_k\}$ by y_k, where $\{\ \}$ denotes the fractional part, the cubature rule is

$$\int_0^1\int_0^1 f(x, y)\,dxdy = \frac{1}{F_N}\sum_{k=1}^{F_N} f(x_k, y_k). \tag{1}$$

The summation can also be taken as running from 0 to $F_N - 1$, which replaces a node 1,0 by 0,0 while leaving the rest unchanged. This cubature rule was also given by Zaremba [5].

The investigators have been interested primarily in the higher-dimensional cases and very little has been published on the two-dimensional case. An examination of the nodes for the two-dimensional case suggested an interesting conjecture about their symmetry properties and a modification which improves the accuracy significantly.