Corollary 7:

$$\sum_{k=0}^{n} p_{0;\,2m}(n-k) \sum_{r} (-1)^r \Delta_{2m-1}(k - \tfrac{1}{2}(3r^2 \pm r)) = \begin{cases} 1 \text{ if } n = 0 \\ (-1)^n 2 \text{ if } n = t^2, \\ \quad \text{for } t = 1, 2, \ldots, \\ 0 \text{ otherwise.} \end{cases}$$

REFERENCES

1. H. L. Alder & Amin A. Muwafi. "Identities Relating the Number of Partitions into an Even and Odd Number of Parts." *The Fibonacci Quarterly* 13 (1975): 337-39.
2. G. H. Hardy & E. M. Wright. *Introduction to the Theory of Numbers*. 4th ed. Oxford: Clarendon Press, 1968.
3. Dean R. Hickerson. "Identities Relating the Number of Partitions into an Even and Odd Number of Parts." *J. Combinatorial Theory* A15 (1973):351-53.
4. Dean R. Hickerson. "Identities Relating the Number of Partitions into an Even and Odd Number of Parts II." *The Fibonacci Quarterly* 16 (1978):5-6.
5. J. J. Sylvester. "A Constructive Theory of Partitions Arranged in Three Acts, an Interact and an Exodion." *Amer. J. Math.* 5 (1882):251-330.

*****

FIBONACCI AND LUCAS NUMBERS OF THE FORMS $w^2 - 1$, $w^3 \pm 1$

NEVILLE ROBBINS
Bernard M. Baruch College, New York, NY 10010

INTRODUCTION

Let F_n and L_n denote the nth Fibonacci and Lucas numbers, respectively. All such numbers of the forms w^2, w^3, $w^2 + 1$ have been determined by J. H. E. Cohn [2], H. London and R. Finkelstein [8], R. Finkelstein [4] and [5], J. C. Lagarias and D. P. Weisser [7], R. Steiner [10], and H. C. Williams [11]. In this article, we find all Fibonacci and Lucas nubmers of the forms $w^2 - 1$, $w^3 \pm 1$.

PRELIMINARIES

(1) $L_n = w^2 \rightarrow n = 1$ or 3
(2) $L_n = 2w^2 \rightarrow n = 0$ or ± 6
(3) $L_n = w^3 \rightarrow n = \pm 1$
(4) $L_n = 2w^3 \rightarrow n = 0$
(5) $L_n = 4w^3 \rightarrow n = \pm 3$
(6) $L_{-n} = (-1)^n L_n$
(7) $(F_n, F_{n-1}) = (L_n, L_{n-1}) = 1$
(8) $3 | F_n$ iff $4 | n$
(9) $L_{2n} = L_n^2 - 2(-1)^n$
(10) $L_{2n+1} = L_n L_{n+1} - (-1)^n$
(11) If $(x, y) = 1$ and $xy = w^n$, then $x = u^n$, $y = v^n$, with $(u, v) = 1$ and $uv = w$.
(12) $F_{4n\pm 1} = F_{2n\pm 1} L_{2n} - 1$
(13) $F_{4n} = F_{2n-1} L_{2n+1} - 1$
(14) $F_{4n-2} = F_{2n-2} L_{2n} - 1$

(15) $F_{4n\pm 1} = F_{2n}L_{2n\pm 1} + 1$
(16) $F_{4n} = F_{2n+1}L_{2n-1} + 1$
(17) $F_{4n-2} = F_{2n-2}L_{2n} + 1$
(18) $L_{m+n} = F_{m-1}L_n + F_m L_{n+1}$
(19) The Diophantine equation $y^2 - D = x^3$, with $y \geq 0$, has precisely the solutions: (-1, 0), (0, 1), (2, 3) if $D = 1$; (1, 2) if $D = 3$; (1, 0) if $D = -1$; no solution if $D = -3$.

Remarks: (1) and (2) are Theorems 1 and 2 in [2]. (3) is Theorem 4 in [8], modified by (6). (4) and (5) follow from Theorem 5 in [7]. (6) through (11) are elementary and/or well known. (12) through (17) appear in Theorem 1 of [3]. (18) is a special case of 1.6, p. 62 in [1]. (19) is excerpted from the tables on pp. 74-75 of [6].

THE MAIN THEOREMS

Theorem 1: $(F_m, L_{m\pm n})|L_n$.

Proof: By (6), it suffices to show that $(F_m, L_{m+n})|L_n$. Let $d = (F_m, L_{m+n})$. (18) $\rightarrow d|F_{m-1}L_n$; (7) $\rightarrow d|L_n$.

Corollary 1: $(F_m, L_{m\pm 2}) = 1$ or 3.

Proof: Let $n = 2$ in Theorem 1.

Corollary 2: $(F_{2n\pm 1}, L_{2n\mp 1}) = 1$.

Proof: (8) $\rightarrow 3 \nmid F_{2n\pm 1}$. The conclusion now follows from Corollary 1.

Lemma 1: Let $(F_i, L_j) = 1$ and $F_i L_j = w^k \neq 0$. Then $k = 2$ implies $j = 1$ or 3; $k = 3$ implies $j = \pm 1$.

Proof: Hypothesis and (11) imply $F_i = u^k$, $L_j = v^k$. The conclusion follows from (1) and (3).

Consider the following equations:

$$\text{(i)}\quad F_m = w^k - 1$$

$$\text{(ii)}\quad F_m = w^k + 1$$

$$\text{(iii)}\quad L_m = w^k - 1$$

$$\text{(iv)}\quad L_m = w^k + 1$$

For given k, a solution is a pair: (m, w). If $|w| \leq 1$, we say the solution is trivial.

Lemma 2: The trivial solutions of (i) through (iv) are as follows:

(i) (0, 1), (-2, 0) for all k; (0, ±1) for k even.
(ii) (±1, 0), (2, 0), (±3, 1) for all k; (0, -1) for k odd.
(iii) (-1, 0) for all k.
(iv) (0, 1), (1, 0) for all k.

Proof: Obvious.

Theorem 2: If $k = 2$, the nontrivial solutions of (i) are (4, 2) and (6, 3).

Proof: *Case 1*.—Let $m = 4n \pm 1$. Hypothesis and (12) $\rightarrow F_{2n\pm 1}L_{2n} = w^2 \neq 0$. Theorem 1 $\rightarrow (F_{2n\pm 1}, L_{2n}) = 1$. Lemma 1 $\rightarrow 2n = 1$ or 3, an impossibility.

Case 2.—Let $m = 4n$. Hypothesis and (13) $\rightarrow F_{2n-1}L_{2n+1} = w^2 \neq 0$.

Case 2.—continued

Corollary 2 and (11) $\to L_{2n-1} = v^2$.
Now (1) $\to 2n + 1 = 1$ or $3 \to n = 0$ or 1.
Hypothesis $\to m \neq 0 \to n \neq 0 \to n = 1 \to m = 4 \to w = 2$.

Case 3.—Let $m = 4n - 2$. Hypothesis and (14) $\to F_{2n}L_{2n-2} = w^2 \neq 0$. Let $d = (F_{2n}, L_{2n-2})$. If $d = 1$, we have a contradiction, as in Case 1. If $d \neq 1$, then Corollary 1 $\to d = 3$. Hence, $(F_{2n}/3)(L_{2n-2}/3) = (w/3)^2$. Now (11) $\to F_{2n} = 3u^2$, $L_{2n-2} = 3v^2$. But $F_{2n} = 3u^2 \to n = 0$ or 2 by a result of R. Steiner [10, pp. 208-10].
Hypothesis $\to m \neq -2 \to n \neq 0 \to n = 2 \to m = 6 \to w = 3$.

Theorem 3: If $k = 3$, then (i) has no nontrivial solution.

Proof: *Case 1.*—Let $m = 4n \pm 1$. As in the proof of Theorem 2, Case 1, we have Lemma 1 $\to 2n = \pm 1$, an impossibility.

Case 2.—Let $m = 4n$. As in the proof of Theorem 2, Case 2, we have $L_{2n+1} = v^3$. Now (3) $\to 2n + 1 = \pm 1 \to n = 0$ or -1.
Hypothesis $\to n \neq 0$ $n = -1 \to m = -4 \to F_{-4} = -3 = w^3 - 1$, an impossibility.

Case 3.—Let $m = 4n - 2$. As in the proof of Theorem 2, Case 3, we have $F_{2n}L_{2n-2} = w^3 \neq 0$, $(F_{2n}, L_{2n-2}) = 3$, so $F_{2n} = 3u^3$, $L_{2n-2} = 3v^3$. Now Theorem 2 of [7] $\to n = 2 \to m = 6 \to F_6 = 8 = w^3 - 1$, an impossibility.

Theorem 4: If $k = 3$, then (ii) has no nontrivial solution.

Proof: *Case 1.*—Let $m = 4n \pm 1$. Hypothesis and (15) $\to F_{2n}L_{2n\pm1} = w^3 \neq 0$. Theorem 1 and Lemma 1 $\to 2n \pm 1 = \pm 1 \to n = 0$ or $\pm 1 \to m = \pm 1, \pm 3, \pm 5$. But $F_{\pm 5} = 5 \neq w^3 + 1$. Therefore, $m = \pm 1, \pm 3$ (trivial solutions).

Case 2.—Let $m = 4n$. Hypothesis and (16) $\to F_{2n+1}L_{2n-1} = w^3 \neq 0$, $n \neq 0$.
Theorem 1 and Lemma 1 $\to 2n - 1 = \pm 1 \to n = 1 \to m = 4 \to F_4 = 3 = w^3 + 1$, an impossibility.

Case 3.—Let $m = 4n + 2$. Hypothesis and (17) $\to F_{2n}L_{2n+2} = w^3 \neq 0$. As in the proof of Theorem 3, Case 3, we have $F_{2n} = 3u^3$, $L_{2n+2} = 3v^3$, an impossibility.

Theorem 5: If $k = 2$, then the nontrivial solutions of (iii) are $(\pm 2, \pm 2)$.

Proof: *Case 1.*—Let $m = 4n$.
Hypothesis and (9) $\to L_{2n}^2 - 2 = w^2 - 1 \to L_{2n}^2 - w^2 = 1 \to L_{2n} - w = L_{2n} + w = \pm 1 \to w = 0 \to L_{2n} = \pm 1$, an impossibility.

Case 2.—Let $m = 4n + 2$.
Hypothesis and (9) $\to L_{2n+1}^2 + 2 = w^2 - 1 \to w^2 - L_{2n+1}^2 = 3 \to L_{2n+1} = \pm 1$, $w = \pm 2$ $m = \pm 2$.

Case 3.—Let $m = 4n + 1$. Hypothesis and (10) $\to L_{2n}L_{2n+1} = w^2$. (7) and (11) $\to L_{2n} = u^2$, $L_{2n+1} = v^2$, contradicting (1).

Case 4.—Let $m = 4n - 1$. Hypothesis and (10) $\to L_{2n}L_{2n-1} + 2 = w^2$.

(9) and (10) $\to \{L_n^2 - 2(-1)^n\}\{L_nL_{n-1} + (-1)^n\} + 2 = w^2$. We have:

$$L_n^3L_{n-1} + (-1)^nL_n^2 - 2(-1)^nL_nL_{n-1} = w^2.$$

Let $M_n = L_n^2L_{n-1} + (-1)^n(L_n - 2L_{n-1})$. Now, $L_nM_n = w^2$. Let p be an

Case 3.—continued

odd prime such that $p^e || L_n$. (7) $\to p \nmid M_n \to p^e || w^2 \to 2 | e$. Therefore, we must have $L_n = u^2$ or $2u^2$.
(1) and (2) $\to n = 0, 1, 3$, or $\pm 6 \to m = -1, 3, 11, 23, -25$. By direct computation of each corresponding L_m, we obtain a contradiction unless $m = -1$ (trivial solution).

Theorem 6: If $k = 3$, then (iii) has the unique nontrivial solution (4, 2).

Proof: *Case 1.*—Let $m = 4n$.
Hypothesis and (9) $\to L_{2n}^2 - 2 = w^3 - 1 \to L_{2n}^2 - 1 = w^3$.
Now (19) $\to L_{2n} = 0, 1$, or $3 \to L_{2n} = 3 \to 2n = 2 \to m = 4 \to w = 2$.

Case 2.—Let $m = 4n + 2$.
Hypothesis and (9) $\to L_{2n+1}^2 + 2 = w^3 - 1 \to L_{2n+1}^2 + 3 = w^3$, contradicting (19).

Case 3.—Let $m = 4n + 1$. Hypothesis and (10) $\to L_{2n}L_{2n+1} = w^3$.
(7) and (11) $\to L_{2n} = u^3$, $L_{2n+1} = v^3$, contradicting (3).

Case 4.—Let $m = 4n - 1$. As in the proof of Theorem 5, Case 4, we have $L_n M_n = w^3$. If p is an odd prime such that $p^e || L_n$, then $p \nmid M_n$, so that $p^e || w^3 \to 3 | e$. Therefore, $L_n = u^3$, $2u^3$, or $4u^3$.

But (3), (4), and (5) $\to n = 0, \pm 1$, or $\pm 3 \to m = -1, 3, -5, 11$, or -13. By direct computation of each corresponding L_m, we obtain a contradiction unless $m = -1$ (trivial solution).

Theorem 7: If $k = 3$, then (iv) has no nontrivial solution.

Proof: *Case 1.*—Let $m = 4n$.
Hypothesis and (9) $\to L_{2n}^2 - 2 = w^3 + 1 \to L_{2n}^2 - 3 = w^3$.
(19) $\to L_{2n} = 2$, $w = 1 \to n = 0 \to m = 0$ (trivial solution).

Case 2.—Let $m = 4n + 2$.
Hypothesis and (9) $\to L_{2n+1}^2 + 2 = w^3 + 1 \to L_{2n+1}^2 + 1 = w^3$.
(19) $\to L_{2n+1} = 0$, $w = 1$, an impossibility.

Case 3.—Let $m = 4n - 1$. Hypothesis and (10) $\to L_{2n}L_{2n-1} = w^3$.
(7) and (11) $\to L_{2n} = u^3$, $L_{2n-1} = v^3$, contradicting (3).

Case 4.—Let $m = 4n + 1$. Hypothesis and (10) $\to L_{2n}L_{2n+1} - 2 = w^3$.
(9) and (10) $\to \{L_n^2 - 2(-1)^n\}\{L_n L_{n+1} - (-1)^n\} - 2 = w^3$. We have:

$$L_n^3 L_{n+1} - (-1)^n L_n^2 - 2(-1)^n L_n L_{n+1} = w^3.$$

Let $M_n = L_n^2 L_{n+1} - (-1)^n (L_n + 2L_{n+1})$. Now, $L_n M_n = w^3$. As in the proof of Theorem 6, Case 4, $n = 0, \pm 1$, or ± 3. Therefore, $m = 1$, $-3, 5, -11, 13$. By direct computation of each corresponding L_m, we obtain a contradiction unless $m = 1$ (trivial solution).

Remark: Cases 1 and 2 could also be disposed of by appeal to Theorem 13 in [9].

SUMMARY OF RESULTS

$$F_m = w^2 - 1 \to w = 0, \pm 1, \pm 2, \pm 3$$
$$F_m = w^3 - 1 \to w = 0, 1$$
$$F_m = w^3 + 1 \to w = -1, 0, 1$$
$$L_m = w^2 - 1 \to w = 0, \pm 2$$
$$L_m = w^3 - 1 \to w = 0, 2$$
$$L_m = w^3 + 1 \to w = 0, 1$$

REFERENCES

1. L. Carlitz and H. H. Ferns. "Some Fibonacci and Lucas Identities." *The Fibonacci Quarterly* 8 (1970):61-73.
2. J. H. E. Cohn. "Square Fibonacci Numbers, Etc." *The Fibonacci Quarterly* 2 (1964):109-13.
3. U. Dudley and B. Tucker. "Greatest Common Divisors in Altered Fibonacci Sequences." *The Fibonacci Quarterly* 9 (1971):89-92.
4. R. Finkelstein. "On Fibonacci Numbers Which Are One More Than a Square." *J. reine angew. Math.* 262/263 (1973):171-82.
5. R. Finkelstein. "On Lucas Numbers Which Are One More Than a Square." *The Fibonacci Quarterly* 13 (1975):340-42.
6. O. Hemer. "Notes on the Diophantine Equation $y^2 - k = x^3$." *Ark. Mat.* 3 (1954):67-77.
7. J. C. Lagarias and D. P. Weisser. "Fibonacci and Lucas Cubes." *The Fibonacci Quarterly* 19 (1981):39-43.
8. H. London and R. Finkelstein. "On Fibonacci and Lucas Numbers Which Are Pecfect Powers." *The Fibonacci Quarterly* 7 (1969):476-81.
9. R. Steiner. "On nth Powers in the Lucas and Fibonacci Series." *The Fibonacci Quarterly* 16 (1978):451-58.
10. R. Steiner. "On Fibonacci Numbers of the Form $x^2 + 1$." In *A Collection of Manuscripts Related to the Fibonacci Sequence: 18th Anniversary Volume*. Santa Clara, Calif.: The Fibonacci Association, 1980, pp. 208-10.
11. H. C. Williams. "On Fibonacci Numbers of the Form $k^2 + 1$." *The Fibonacci Quarterly* 13 (1975):213-14.

*****

THE ANDREWS FORMULA FOR FIBONACCI NUMBERS

MICHAEL D. HIRSCHHORN

University of New South Wales, Sydney, Australia

1. George E. Andrews [1] gave the following formulas for the Fibonacci numbers F_n ($F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$) in terms of binomial coefficients $\left((n;\ r) = \binom{n}{r}\right)$:

$$F_n = \sum_j (-1)^j \quad (n - 1;\ [(n - 1 - 5j)/2]), \tag{1.1}$$

$$F_n = \sum_j (-1)^j \quad (n;\ [(n - 1 - 5j)/2]). \tag{1.2}$$

Hansraj Gupta [2] has pointed out that (1.1) and (1.2) can be written, respectively, as

$$F_{2m+1} = S(2m,\ m) - S(2m,\ m - 2), \tag{1.3a}$$

$$F_{2m+2} = S(2m + 1,\ m) - S(2m + 1,\ m - 2) \tag{1.3b}$$

and

$$F_{2m+1} = S(2m + 1,\ m) - S(2m + 1,\ m - 1) \tag{1.4a}$$

$$F_{2m+2} = S(2m + 2,\ m) - S(2m + 1,\ m - 1), \tag{1.4b}$$

where $S(n,\ k) = \Sigma(n;\ j)$, the sum being taken over those j congruent to k modulo 5, and has given inductive proofs of (1.3) and (1.4).