SUMMATION OF SECOND-ORDER RECURRENCE TERMS AND THEIR SQUARES*

DAVID L. RUSSELL
Bell Laboratories, Holmdel, NJ 07733

Consider the linear recurrence sequence $\{R_n\}$ defined by $R_n = pR_{n-1} + qR_{n-2}$ for all n, where p and q are real. Initial conditions for any two consecutive terms completely define the sequence. We are interested in finding sums of the form

$$\sum_{x \le i \le y} R_i \quad \text{and} \quad \sum_{x \le i \le y} R_i^2$$

for arbitrary values of p and q.

Theorem 1:

$$\sum_{x \le i \le y} R_i = \left[\frac{1}{p+q-1}(qR_n + R_{n+1})\right]_{n=x-1}^{n=y}, \qquad \text{if } p+q-1 \ne 0. \tag{1}$$

$$\sum_{x \le i \le y} R_i = \left[\frac{1}{q+1}(qR_n + n(qR_0 + R_1))\right]_{n=x-1}^{n=y}, \quad \text{if } p+q-1 = 0,\ q+1 \ne 0. \tag{2}$$

$$\sum_{x \le i \le y} R_i = \left[\frac{n}{2}(R_n + R\ \ )\right]_{n=x-1}^{n=y}, \qquad \text{if } p+q-1 = 0,\ q+1 = 0. \tag{3}$$

The solution to the recurrence relation is determined by the roots of the characteristic equation $x^2 - px - q = 0$ and by the initial conditions.

If the two roots α and β of the characteristic equation are distinct and different from 1, then the solution of the recurrence is $R_n = a\alpha^n + b\beta^n$, where a and b are constants determined by the initial conditions. The sum may be calculated easily from the formula for the sum of a geometric series and from the equation

$$(\alpha - 1)(q + \alpha) = (p + q - 1)\alpha. \tag{4}$$

If α is a double root of the characteristic equation and $\alpha \ne 1$, then the solution of the recurrence is $R_n = a\alpha^n + bn\alpha^n$, where again a and b are constants determined by the initial constants. Multiplying (4) by $\alpha^n/(\alpha - 1)$, and taking the derivative with respect to α gives the following equation:

$$qn\alpha^{n-1} + (n+1)\alpha^n = \frac{(p+q-1)[n\alpha^{n+1} - (n+1)\alpha^n]}{(\alpha - 1)^2}; \tag{5}$$

the appropriate summation formula can be simplified with (5) to give (1).

Equations (2) and (3) apply to the degenerate cases where the roots of the characteristic equation are $(p - 1, 1)$ and $(1, 1)$, respectively. The corresponding summations have nongeometric terms in them and simplify to different forms.

The results of Theorem 1 are well known, particularly equation (1) (see, for example, [2] and [3]). Often, however, the need for separate proofs for the cases of a double root and a root equal to 1 is not recognized. In the special case that $p = q = 1$, equation (1) applies, and we have, as simple corollaries, formulas for the summation of Fibonacci and Lucas numbers:

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$$\sum_{1 \le i \le n} F_i = F_{n+2} - 1, \qquad \sum_{1 \le i \le n} L_i = L_{n+2} - 3.$$

In Theorem 1 the results depend on the roots of the characteristic equation. If we consider the sum of the *squares* of the recurrence terms, the results depend on the possible values for the *products* of two roots.

Theorem 2: Let $\{R_n\}$ satisfy

$$R_n = pR_{n-1} + qR_{n-2}$$

and let $\{S_n\}$ satisfy

$$S_n = pS_{n-1} + qS_{n-2}$$

for all n and all real p, q.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{q^2(1-q)R_nS_n + pqR_nS_{n+1} + pqR_{n+1}S_n + (1-q)R_{n+1}S_{n+1}}{(q+1)(p+q-1)(p-q+1)}\right]_{n=x-1}^{n=y}, \tag{6}$$

if $q + 1 \ne 0$, $p + q - 1 \ne 0$, $p - q + 1 \ne 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{q^2}{q^2 - 1} R_n S_n - \frac{q}{q^2 - 1}(bS_n + dR_n) + bdn\right]_{n=x-1}^{n=y}, \tag{7}$$

where $b = (qR_0 + R_1)/(q + 1)$, $d = (qS_0 + S_1)/(q + 1)$,
if $q + 1 \ne 0$, $p + q - 1 = 0$, $p - q + 1 \ne 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{q^2}{q^2 - 1} R_n S_n - \frac{q}{q^2 - 1}(-1)^n(bS_n + dR_n) + bdn\right]_{n=x-1}^{n=y}, \tag{8}$$

where $b = (qR_0 - R_1)/(q + 1)$, $d = (qS_0 - S_1)/(q + 1)$,
if $q + 1 \ne 0$, $p + q - 1 \ne 0$, $p - q + 1 = 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{R_0S_0 + R_1S_1}{2} n + \frac{R_0S_0 - R_1S_1}{2} \cdot \frac{(-1)^n}{2}\right]_{n=x-1}^{n=y}, \tag{9}$$

if $q + 1 \ne 0$, $p + q - 1 = 0$, $p - q + 1 = 0$.

$$\sum_{x \le i \le y} R_i S_i = \frac{1}{(\alpha^2 - 1)^2}\left[ac\frac{\alpha^{2n+2}}{\alpha^2 - 1} + (bc + ad)n + bd\frac{1}{1 - \alpha^2}\left(\frac{1}{\alpha}\right)^{2n}\right]_{n=x-1}^{n=y}, \tag{10}$$

where $\alpha = \frac{1}{2}(p + (p^2 - 4)^{\frac{1}{2}})$, and
$a = (\alpha R_1 - R_0)$,
$b = \alpha(\alpha R_0 - R_1)$,
$c = (\alpha S_1 - S_0)$,
$d = \alpha(\alpha S_0 - S_1)$,
if $q + 1 = 0$, $p + q - 1 \ne 0$, $p - q + 1 \ne 0$.

$$\sum_{x \le i \le y} R_i S_i = \left[R_0S_0 + \tfrac{1}{2}(R_1S_1 - R_{-1}S_{-1})\frac{n(n+1)}{2} + (R_1 - R_0)(S_1 - S_0)\frac{n(n+1)(2n+1)}{6}\right]_{n=x-1}^{n=y}, \tag{11}$$

if $q + 1 = 0$, $p + q - 1 = 0$, $p - q + 1 \ne 0$.

$$\sum_{x\le i\le y} R_iS_i = \left[R_0S_0 + \tfrac{1}{2}(R_1S_1 - R_{-1}S_{-1})\frac{n(n+1)}{2} + (R_1+R_0)(S_1+S_0)\frac{n(n+1)(2n+1)}{6}\right]_{n=x-1}^{n=y}, \tag{12}$$

if $q + 1 = 0$, $p + q - 1 \neq 0$, $p - q + 1 = 0$.

Proof: The key relation, analogous to (4), is the following, where α and β are roots of $x^2 - px - q = 0$, $\alpha\beta \neq 1$, $\alpha^2 \neq 1$, $\beta^2 \neq 1$:

$$\frac{\alpha^{n+1}\beta^{n+1}}{\alpha\beta - 1} = \frac{q^2(1-q)\alpha^n\beta^n + pq\alpha^n\beta^{n+1} + pq\alpha^{n+1}\beta^n + (1-q)\alpha^{n+1}\beta^{n+1}}{(q+1)(p-q+1)(p+q-1)}. \tag{13}$$

This is proved by considering the following equation (recall that $\alpha^2 = p\alpha + q$ and $\beta^2 = p\beta + q$):

$$\begin{aligned}(\alpha\beta - 1)&[q^2(1-q) + pq\beta + pq\alpha + (1-q)\alpha\beta]\\ &= \alpha\beta q^2(1-q) + \alpha\beta^2 pq + \alpha^2\beta pq + \alpha^2\beta^2(1-q) - q^2(1-q) - pq\beta - pq\alpha - (1-q)\alpha\beta\\ &= \alpha\beta q^2(1-q) + \alpha(p\beta+q)pq + (p\alpha+q)\beta pq + (p\alpha+q)(p\beta+q)(1-q)\\ &\qquad - q^2(1-q) - pq\beta - pq\alpha - (1-q)\alpha\beta\\ &= \alpha\beta[q^2(1-q) + p^2q + p^2q + p^2(1-q) - (1-q)]\\ &= \alpha\beta[p^2(q+1) - (q^2-1)(q-1)]\\ &= \alpha\beta(q+1)(p+q-1)(p-q+1).\end{aligned} \tag{14}$$

Now $\alpha\beta = 1$ is possible if and only if (1) $p + q - 1 = 0$ (the roots are $p - 1$ and 1); (2) $p - q + 1 = 0$ (the roots are $p + 1$ and -1); or (3) $q + 1 = 0$ (the roots are reciprocals). Thus we can divide both sides of (14) by

$$(\alpha\beta - 1)(q + 1)(p + q - 1)(p - q + 1);$$

multiplying by $\alpha^n\beta^n$ gives (13).

In the remainder of the proof, we use α and β to represent roots of

$$x^2 - px - q = 0,$$

we use a, b, c, d to represent constants determined by initial conditions of the recurrences, and we let

$$\Delta = (q + 1)(p + q - 1)(p - q + 1).$$

If omitted, the limits of summation are understood to be x and y; the right-hand sides are to be evaluated at $n = y$ and $n = x - 1$.

Suppose that $\alpha \neq \beta$. Then the solutions to the recurrences are

$$R_n = a\alpha^n + b\beta^n \quad\text{and}\quad S_n = c\alpha^n + d\beta^n.$$

$$\begin{aligned}\Delta\Sigma R_iS_i &= \Delta\Sigma(a\alpha^i + b\beta^i)(c\alpha^i + d\beta^i)\\ &= \Delta\Sigma(ac\alpha^{2i} + ad\alpha^i\beta^i + bc\alpha^i\beta^i + bd\beta^{2i})\\ &= \Delta\left[\frac{ac\alpha^{2n+2}}{\alpha^2 - 1} + \frac{ad(\alpha\beta)^{n+1}}{\alpha\beta - 1} + \frac{bc(\alpha\beta)^{n+1}}{\alpha\beta - 1} + \frac{bd\beta^{2n+2}}{\beta^2 - 1}\right].\end{aligned}$$

Since $q + 1 \neq 0$, $p + q - 1 \neq 0$, and $p - q + 1 \neq 0$, we know that $\alpha^2 \neq 1$, $\beta^2 \neq 1$, and $\alpha\beta \neq 1$. Equation (13) can thus be applied to each term individually; when terms are collected the desired result is obtained:

$$\begin{aligned}\Delta\Sigma R_iS_i &= q^2(1-q)[ac\alpha^n\alpha^n + ad\alpha^n\beta^n + bc\alpha^n\beta^n + bd\beta^n\beta^n]\\ &\quad + pq[ac\alpha^n\alpha^{n+1} + ad\alpha^n\beta^{n+1} + bc\alpha^n\beta^{n+1} + bd\beta^n\beta^{n+1}]\\ &\quad + pq[ac\alpha^{n+1}\alpha^n + ad\alpha^{n+1}\beta^n + bc\alpha^{n+1}\beta^n + bd\beta^{n+1}\beta^n]\\ &\quad + (1-q)[ac\alpha^{n+1}\alpha^{n+1} + ad\alpha^{n+1}\beta^{n+1} + bc\alpha^{n+1}\beta^{n+1} + bd\beta^{n+1}\beta^{n+1}]\\ &= q^2(1-q)R_nS_n + pqR_nS_{n+1} + pqR_{n+1}S_n + (1-q)R_{n+1}S_{n+1}.\end{aligned}$$

If α is a double root of $x^2 - px - q = 0$, then the sum takes the following form:

$$\begin{aligned}\Delta\Sigma R_i S_i &= \Delta\Sigma(a\alpha^i + bi\alpha^i)(c\alpha^i + di\alpha^i) \\ &= \Delta\Sigma(ac\alpha^{2i} + adi\alpha^{2i} + bci\alpha^{2i} + bdi^2\alpha^{2i}).\end{aligned} \tag{15}$$

By taking various derivatives of (13), it is easy to show that the following expressions hold:

$$\begin{aligned}\Delta\Sigma i\alpha^i\beta^i &= q^2(1-q)n\alpha^n\beta^n + pq(n+1)\alpha^n\beta^{n+1} + pqn\alpha^{n+1}\beta^n + (1-q)(n+1)\alpha^{n+1}\beta^{n+1} \\ &= q^2(1-q)n\alpha^n\beta^n + pqn\alpha^n\beta^{n+1} + pq(n+1)\alpha^{n+1}\beta^n + (1-q)(n+1)\alpha^{n+1}\beta^{n+1},\end{aligned}$$

$$\begin{aligned}\Delta\Sigma i^2\alpha^i\beta^i &= q^2(1-q)n^2\alpha^n\beta^n + pqn(n+1)\alpha^n\beta^{n+1} + pqn(n+1)\alpha^{n+1}\beta^n \\ &\quad + (1-q)(n+1)^2\alpha^{n+1}\beta^{n+1}.\end{aligned}$$

Substitution into (15) and simplification complete the proof of (6).

Equations (7-12) apply in various degenerate cases where the product of some two roots of the characteristic equation is 1, and there is a nongeometric term in the corresponding summation:

- in equation (7) the roots are $(\alpha, 1)$, $\alpha \neq 1, -1$;
- in equation (8) the roots are $(\alpha, -1)$, $\alpha \neq 1, -1$;
- in equation (9) the roots are $(1, -1)$;
- in equation (10) the roots are (α, α^{-1});
- in equation (11) the roots are $(1, 1)$;
- in equation (12) the roots are $(-1, -1)$.

The results of Theorem 2 correct and complete the discussion of Hoggatt [1]. Note that if $q = 1$ and $p \neq 0$ the following special cases are derived (see also Russell [5]):

$$\sum_{x \le i \le y} R_i = \left[\frac{R_n + R_{n+1}}{p}\right]_{n=x-1}^{n=y},$$

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{R_{n+1}S_n + R_n S_{n+1}}{2p}\right]_{n=x-1}^{n=y}.$$

Nothing in the derivations has precluded the possibility that $q = 0$. In this case the recurrences are first-order recurrences and the solutions are readily seen to reduce to the appropriate sums.

The method of this paper can be extended to other sums involving products of terms from recurrence sequences. The "most pleasing" sums derived are those that can be expressed as linear combinations of terms "similar" to the summand, without multiplications by functions of n. Such sums, as in equations (1) and (6), have been called *standard sums* in [4], where they are more precisely defined. It seems clear from the proofs of Theorems 1 and 2 that such standard sums do not exist if there is a set of values $\{\alpha_i | \alpha_i$ is a root of the characteristic equation of the ith recurrence sequence in the product being summed$\}$ such that $\Pi\alpha_i = 1$. When such a standard sum does exist, it can be found directly, without knowing the roots of the characteristic equations, by the method described in [4]. The "key formulas" (4) and (13) were, in fact, first found in this way.

The sums found are, of course, not unique. For instance, using the relation

$$R_{n+2}S_{n+2} = p^2R_{n+1}S_{n+1} + pqR_{n+1}S_n + pqR_nS_{n+1} + q^2R_nS_n,$$

equation (6) can also be written as follows:

$$\sum_{x \le i \le y} R_i S_i = \left[\frac{pq^2(R_n S_{n+1} + R_{n+1}S_n) + (1 - q)[R_{n+2}S_{n+2} + (1 - p^2)R_{n+1}S_{n+1}]}{(1 - q)(p + q - 1)(p - q + 1)}\right]_{n=x-1}^{n=y},$$

$$\text{if } q + 1 \ne 0,\ p + q - 1 \ne 0,\ p - q + 1 \ne 0. \tag{16}$$

In closing, we note that the expressions of this paper can be used to derive some identities among recurrence terms. As an example consider $\Sigma R_i S_i$ with R_i and S_i identical sequences, $R_0 = S_0 = 0$, $R_1 = S_1 = 1$, $p = 1$, $q = 2 + \varepsilon$, and limits of summation $0 \le i \le n$. As $\varepsilon \to 0$, the sum approaches a well-defined value, and thus the right-hand side of (16) must also have a finite limit. Since the denominator goes to 0, so must the numerator. We conclude that the following must be true:

$$\left[8R_y R_{y+1} - R_{y+2}^2\right]_{y=-1}^{y=n} = 8R_n R_{n+1} - R_{n+2}^2 + 1 = 0$$

or

$$8R_n R_{n+1} = (R_{n+2} + 1)(R_{n+2} - 1)$$

$$\text{if } p = 1,\ q = 2,\ R_0 = 0,\ R_1 = 1.$$

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ITERATING THE PRODUCT OF SHIFTED DIGITS

SAMUEL S. WAGSTAFF, JR.
Northern Illinois University, DeKalb, IL 60115

1. INTRODUCTION

Let t be a fixed nonnegative integer. For positive integers n written in decimal as

$$n = \sum_{i=0}^{k} d_i \cdot 10^i,$$

with $0 \le d_i \le 9$ and $d_k > 0$, we define

$$f_t(n) = \prod_{i=0}^{k} (t + d_i).$$

Also define $f_0(0) = 0$. Erdös and Kiss [1] have asked about the behavior of the sequence of iterates n, $f_t(n)$, $f_t(f_t(n))$, ... They noted that $f_4(120) = 120$. For $t = 0$, every such sequence eventually reaches a one-digit number. Sloane