and so

$$B(n,\ 0) = \frac{(-1)^n}{(n-1)!}\sum_{k=1}^{n-1}\frac{1}{k}.$$

Since

$$\sum_{k=1}^{m}\frac{1}{k} = \sum_{k=1}^{m}\frac{(-1)^{k-1}}{k}\binom{m}{k},$$

we see that each $B(n,\ 0)$ may be regarded as a binomial sum.

On the other hand,

$$f_0(x) = (1-x)^{-1} = \sum_{k=0}^{\infty} x^k$$

and term by term integration of this power series gives

$$f_n(x) = x^n\sum_{k=0}^{\infty}\frac{x^k}{(k+1)\cdot\cdots\cdot(k+n)}.$$

For $n \geq 2$, this series converges at $x = \pm 1$ and is uniformly convergent on the closed interval $[-1,\ 1]$. By Abel's theorem for power series, the values of our functions at the endpoints of the interval of convergence are given by the power series

$$\lim_{x\uparrow 1} f_n(x) = \sum_{k=0}^{\infty}\frac{1}{(k+1)\cdot\cdots\cdot(k+n)} = \frac{1}{n!}\sum_{k=0}^{\infty}\binom{n+k}{k}^{-1} = \frac{1}{n!}\cdot\frac{n}{n-1},$$

by out Theorem 2, while our Theorem 3 gives

$$\lim_{x\downarrow -1} f_n(x) = (-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)\cdot\cdots\cdot(k+n)} = \frac{(-1)^n}{n!}\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}^{-1} = \frac{(-1)^n}{n!}J_n.$$

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TILING THE PLANE WITH INCONGRUENT REGULAR POLYGONS

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Professor Michael Edelstein asked me how to tile the Euclidean plane with squares of integer side lengths all of which are incongruent. The question can be answered in a way that involves a perfect squared square and a geometric application of the Fibonacci numbers.

A perfect squared square is a square of integer side length which is tiled with more than one (but finitely many) component squares of integer side lengths all of which are incongruent. For more information, see the survey articles [3] and [5]. A perfect squared square is simple if it contains no proper subrectangle formed from more than one component square; otherwise it is compound. It is known ([3], p. 884) that a compound perfect squared square must have at least 22 components. Duijvestijn's simple perfect squared square [2] (see Fig. 1) thus has the least possible number of components (21).

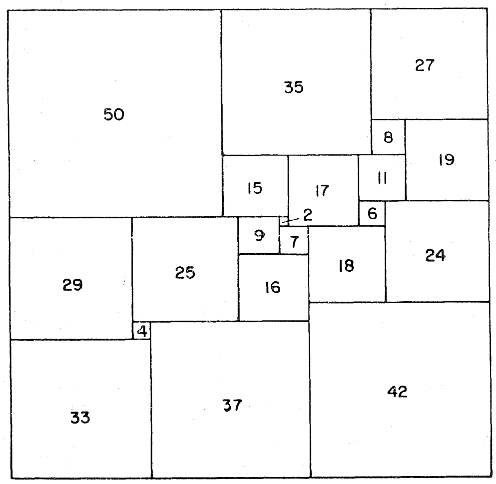

FIGURE 1

The Fibonacci numbers are defined recursively by $f_1 = 1$, $f_2 = 1$, and

(*) $$f_{n+2} = f_n + f_{n+1} \qquad (n \geq 1).$$

They are used in connection with the tiling shown in Figure 2. Its nucleus is a 21 component Duijvestijn square, indicated by diagonal hatching, having side length $s = f_1 \cdot s = 112$, as in Figure 1.

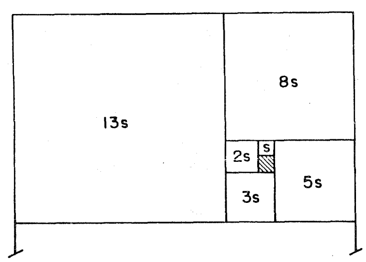

FIGURE 2

On top of this square we tile a one-component square s of side length $f_2 \cdot s = s = 112$, forming an overall rectangle of dimensions $2s$ by s. On the left side of this rectangle (the longer edge) we tile a square $2s$ of side length $f_3 \cdot s = 2s = 224$, forming an overall rectangle of dimensions $3s$ by $2s$. We now proceed counterclockwise as shown, each time tiling a square $f_n s$ onto the required longer edge of the last overall rectangle of dimensions $f_n s$ by $f_{n-1}s$, forming a new overall rectangle of dimensions $f_{n+1}s$ by $f_n s$—this follows from (*). The tiling can continue indefinitely in this way at each stage, because $f_n s = f_{n-1}s + f_{n-4}s + f_{n-3}s$ [this is used for $n \geq 5$ and also follows from (*)]. A closely related Fibonacci tiling for a single quadrant of the plane (but beginning with two congruent squares) occurs in [1, p. 305, Fig. 3].

If we consider the center of the nuclear hatched square as the origin, O, of the plane, it is clear that the tiling eventually covers an arbitrary disc centered at O and thus covers the whole plane. Finally, note that all the component squares used in the tiling have integer side lengths and are incongruent.

The tiling described above may be called static, since the tiles remain fixed where placed, and the outward growth occurs at the periphery. It is also interesting to consider a dynamic tiling. Start with a Duijvestijn square. Its smallest component has side length 2. Enlarge it by a factor of 56. The smallest component in the resulting square has side length 112. Replace it by a Duijvestijn square. Now enlarge the whole configuration again by a factor of 56. Repeat this process indefinitely, thus obtaining the tiling. Here no tile remains fixed, outward growth occurs everywhere, and it is impossible to write down a sequence of side lengths of squares used in the tiling.

The three-dimensional version of this tiling problem (due to D. F. Daykin) is still unsolved: Can 3-space be filled with cubes, all with integer side lengths, no two cubes being the same size? ([4], p. 11).

The plane can also be tiled with incongruent regular triangles and a single regular hexagon, all having integer side lengths.

Begin with regular hexagon I (see Fig. 3) and tile regular triangles with side lengths 1, 2, 3, 4, and 5 counterclockwise around it as shown. Now tile a regular triangle with side length 7 along the sixth side of the hexagon. This counterclockwise tiling can be continued indefinitely to cover the plane. The recursion formula for the side lengths of the triangles is

$$s_i = i \text{ for } 1 \leq i \leq 5,\ s_6 = 7,\ s_i = s_{i-1} + s_{i-5} \text{ for } i \geq 7.$$

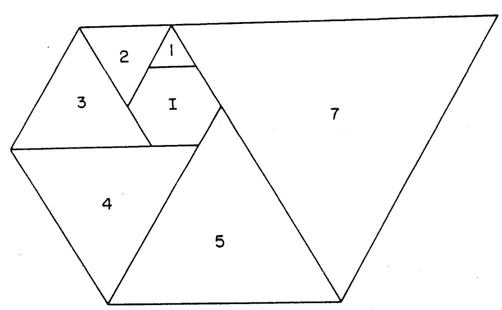

FIGURE 3

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