ADVANCED PROBLEMS AND SOLUTIONS

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Send all communications concerning Advanced Problems and Solutions to Verner E. Hoggatt, Jr., Mathematics Department, San Jose State College, San Jose, California. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-29 *Proposed by Brother U. Alfred, St. Mary's College, California.*

Find the value of a satisfying the relation

$$n^n + (n + a)^n = (n + 2a)^n$$

in the limit as n approaches infinity.

H-30 *Proposed by J. A. H. Hunter, Toronto, Ontario, Canada*

Find all non-zero integral solutions to the two Diophantine equations,

(a) $$X^2 + XY + X - Y^2 = 0$$

(b) $$X^2 - XY - X - Y^2 = 0 \quad .$$

H-31 *Proposed by Verner E. Hoggatt, Jr., San Jose State College, San Jose, California*

Prove the following:

Theorem: Let a, b, c, d be integers satisfying $a > 0$, $d > 0$ and $ad - bc = 1$, and let the roots of $\lambda^2 - \lambda - 1 = 0$ be the fixed points of

$$W = \frac{az + b}{cz + d} \quad .$$

Then it is necessary and sufficient for all integral $n \neq 0$, that $a = F_{2n+1}$, $b = c = F_{2n}$, and $d = F_{2n-1}$, where F_n is the n^{th} Fibonacci number. ($F_1 = 1$, $F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all integral n.)

H-32 *Proposed by R. L. Graham, Bell Telephone Laboratories, Murray Hill, N. J.*

Prove the following:

Given a positive integer n, if there exist m line segments L_i having lengths a_i, $1 \leq a_i \leq n$, for all $1 \leq i \leq m$, such that no three L_i can be used to form a non-degenerate triangle then $F_m \leq n$, where F_m is the m^{th} Fibonacci number.

H-33 *Proposed by Malcolm Tallman, Brooklyn, N. Y.*

If a Lucas number is a prime number and its subscript is composite, then the subscript must be of the form 2^m, $m \geq 2$.

SOLUTIONS

A TOUGH PROBLEM

H-1 *Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.*

Find a formula for the n^{th} non-Fibonacci number, that is, for the sequence $4, 6, 7, 9, 10, 11, 12, 14, 15, 16, 17, 18, 19, 20, 22, 23, \cdots$. (See paper by L. Moser and J. Lambek, American Mathematical Monthly, vol. 61 (1954), pp. 454-458.)

A paper by the proposer will soon appear in the Fibonacci Quarterly, which will discuss this problem.

A WORLD-FAMOUS PROBLEM

H-2 *Proposed by L. Moser and L. Carlitz, University of Alberta, Edmonton, Alberta, and Duke University, Durham, N. C.*

Resolve the conjecture: There are no Fibonacci numbers which are integral squares except 0, 1, and 144.

See "Lucas Squares," by Brother U. Alfred in this issue. A discussion by J. H. E. Cohn on Fibonacci Squares will be in the next issue.

AN UNSOLVED PROBLEM

H-15 *Proposed by Malcolm H. Tallman, Brooklyn, N. Y.*

Do there exist integers N_1, N_2, and N_3 for which the following expressions cannot equal other Fibonacci numbers?

(i) $F_n^3 - F_n^2 F_m - F_m^3 \qquad m, n \geq N_1$,

(ii) $F_n^3 + F_n^2 F_m + F_n F_m^2 \qquad m, n \geq N_2$,

(iii) $F_n^2 - 3F_m^3 \qquad m, n \geq N_3$.

No discussion of any kind has been received on this problem.

AN INSPIRING PROBLEM

H-17 *Proposed by Brother U. Alfred, St. Mary's College, California*

Sum
$$\sum_{k=1}^{n} k^3 F_k$$

(Editorial Comment: There will be three different approaches to the solution of the general case of the above problem which will appear soon in the Fibonacci Quarterly.)

Solution by Joseph Erbacher and John Allen Fuchs, University of Santa Clara, Calif.

Let $L(E) = (E^2 - E - 1)^4 = \sum_{i=0}^{8} a_i E^i$ where E is the linear operator such that $E^i F_k = F_{k+i}$. Then $L(E)k^3F_k = 0$. (This follows from a result of James A. Jeske, "Linear Recurrence Relations – Part I," Fibonacci Quarterly, April, 1963, p. 72, Equation (4.8).) Let $S = \sum_{k=1}^{n} k^3F_k$. Since $\sum_{i=0}^{8} a_i = 1$, $S = \sum_{i=0}^{8} a_i S = \sum_{i=0}^{8} \left\{ a_i \sum_{k=i+1}^{n+i} k^3F_k + a_i \sum_{j=1}^{i} [j^3F_j - (n+j)^3 F_{n+j}] \right\} = R + T$, where R is the first double summation and T is the second double summation. Reversing the order of summation in R, we have $R = \sum_{k=1}^{n} \sum_{i=0}^{8} a_i(i+k)^3 F_{i+k}$. Since $\sum_{i=0}^{8} a_i(i+k)^3F_{i+k} = \sum_{i=0}^{8} a_i E^i k^3F_k = L(E)k^3F_k = 0$, it follows that $R = 0$ and $S = T$. Using the relation $F_{n+2} = F_{n+1} + F_n$ in T, one can transform the solution into the form $S = 50 + (n^3 + 6n = 12)F_{n+2} + (-3n^2 + 9n = 19)F_{n+3}$.

Generalizing on the above technique one sees that $\sum_{k=1}^{n} k^p F_k = u(n)F_{n+2}$

$+ v(n)F_{n+3} + A_p$, where u and v are polynomials in n of degree p and A_p is a constant independent of n. It can be shown that the coefficients of u and v may be found by solving the $2p + 2$ equations obtained by letting n take on any $2p + 2$ consecutive values.

Also solved by Zvi Dresner and Marjorie Bicknell

A CLASSICAL SOLUTION

H-16 *Proposed by H. W. Gould, West Virginia University, Morgantown, W. Va.*

Define the ordinary Hermite polynomials by $H_n = (-1)^n e^{x^2} D^n (e^{-x^2})$.

Show that:

(i) $$\sum_{n=0}^{\infty} H_n(x/2)\frac{x^n}{n!} = 1 ,$$

(ii) $$\sum_{n=0}^{\infty} H_n(x/2)\frac{x^n}{n!}F_n = 0 ,$$

(iii) $$\sum_{n=0}^{\infty} H_n(x/2)\frac{x^n}{n!}L_n = 2e^{-x^2} ,$$

where F_n and L_n are the n^{th} Fibonacci and n^{th} Lucas numbers, respectively.

We recall that $\sum_{n=0}^{\infty} H_n(t)\frac{x^n}{n!} = e^{2tx-x^2}$. For $t = \frac{x}{2}$ this reduces to $\sum_{n=0}^{\infty} H_n\left(\frac{x}{2}\right)\frac{x^n}{n!} = 1$.

Put $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Then $(\alpha - \beta)\sum_{n=0}^{\infty} H_n\left(\frac{x}{2}\right)\frac{x^n}{n!}F_n = e^{(\alpha-\alpha^2)x^2} - e^{(\beta-\beta^2)x^2} = 0$ since $\alpha - \alpha^2 = \beta - \beta^2 = -1$.

Similarly,

$$\sum_{n=0}^{\infty} H_n\left(\frac{x}{2}\right)\frac{x^n}{n!}L_n = e^{(\alpha-\alpha^2)x^2} + e^{(\beta-\beta^2)x^2} = 2e^{-x^2} .$$

See also the solution in the last issue by Zvi Dresner.

Reference continued from page 44.

1. K. F. Roth, "Rational Approximations to Algebraic Numbers," Mathematika 2 (1955) pp. 1 – 20, p. 168.