Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Mathematics Department, University of Santa Clara, Santa Clara, California. We welcome any problems believed to be new in the area of recurrent sequences as well as new approaches to existing problems. The proposer should submit his problem with solution in legible form, preferably typed in double spacing, with name(s) and address of the proposer clearly indicated.

Solutions to problems listed below should be submitted within two months of publication.

B-30 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, Calif.

Find the millionth term of the sequence \( a_n \) given that

\[ a_1 = 1, \ a_2 = 1, \ \text{and} \ a_{n+2} = a_{n+1} - a_n \ \text{for} \ n \geq 1. \]

B-31 Proposed by Douglas Lind, Falls Church, Virginia

If \( n \) is even, show that the sum of \( 2n \) consecutive Fibonacci numbers is divisible by \( F_n \).

B-32 Proposed by Charles R. Wall, Texas Christian University, Ft. Worth, Texas.

Show that \( nL_n \equiv F_n \pmod{5} \).

B-33 Proposed by John A. Fuchs, University of Santa Clara, Santa Clara, California

Let \( u_n, v_n, \cdots, w_n \) be sequences each satisfying the second order recurrence formula

\[ y_{n+2} = gy_{n+1} + hy_n \ (n \geq 1), \]

where \( g \) and \( h \) are constants. Let \( a, b, \cdots, c \) be constants. Show that
is true for all positive integral values of \( n \) if it is true for \( n = 1 \) and \( n = 2 \).

B-34 Proposed by G. L. Alexanderson, University of Santa Clara, Santa Clara, California

Let \( u_n \) and \( v_n \) be any two sequences satisfying the second-order recurrence formula

\[
y_{n+2} = gy_{n+1} + hy_n
\]

where \( g \) and \( h \) are constants. Show that the sequence of products \( w_n = u_n v_n \)
satisfies a third-order recurrence formula

\[
y_{n+3} = ay_{n+2} + by_{n+1} + cy_n
\]

and find \( a, b, \) and \( c \) as functions of \( g \) and \( h \).

B-35 Proposed by J. L. Brown, Jr., Pennsylvania State University, University Park, Pa.

Prove that

\[
\sum_{k=1}^{r-1} (-1)^k \binom{r}{k} F_k = 0
\]

for \( r \) an odd positive integer and generalize.

B-36 Proposed by Roseanna Torretto, University of Santa Clara, Santa Clara, California

The sequence 1, 2, 5, 12, 29, 70, \ldots is defined by \( c_1 = 1, c_2 = 2, \) and \( c_{n+2} = 2c_{n+1} + c_n \) for all \( n \geq 1 \). Prove that \( c_{5m} \) is an integral multiple of 29 for all positive integers \( m \).

B-37, Proposed by Brother V. Alfred, St. Mary's College, California

Given a line with a point of origin \( O \) and four positive positions \( A, B, C, \) and \( D \) with respect to \( O \). If the line segments \( OA, OB, OC, \) and \( OD \) correspond respectively to four consecutive Fibonacci numbers \( F_n, F_{n+1}, F_{n+2}, F_{n+3} \)
determine for which set(s) of Fibonacci numbers the points \( A, B, C, \) and \( D \)
are in simple harmonic ratio, i.e.,

\[
\frac{AB}{AD} = \frac{BC}{DC} = -1.
\]
SOLUTIONS

DIFFERENCES MADE INTO PRODUCTS

B-17 Proposed by Charles R. Wall, Ft. Worth, Texas

If \( m \) is an integer, prove that

\[
F_{n+4m+2} - F_n = L_{2m+1} F_{n+2m+1}
\]

where \( F_p \) and \( L_p \) are the \( p \)th Fibonacci and Lucas numbers, respectively.

Solution by I. D. Ruggles, San Jose State College, San Jose, California

In "Some Fibonacci Results Using Fibonacci-Type Sequences," Fibonacci Quarterly, Vol. 1, No. 2, p. 77, it is shown that

\[
F_{q+p} - F_{q-p} = L_p F_q \quad \text{for } p \text{ odd.}
\]

If \( q = n + 2m + 1 \) and \( p = 2m + 1 \), then this becomes the desired formula.

Also solved by Douglas Lind, Falls Church, Virginia, and the proposer.

A TRIGONOMETRIC SUM

B-18 Proposed by J. L. Brown, Jr., Pennsylvania State University, State College, Pennsylvania.

Show that

\[
F_n = 2^{n-1} \sum_{k=0}^{n-1} (-1)^k \cos \frac{n-k-1}{5} \pi \sin \frac{k}{10} \pi \quad \text{for } n \geq 0.
\]

(It should be "for \( n \geq 1 \)" instead of "for \( n \geq 0 \)."

Solution by the proposer

It is well known (e.g., I. J. Schwatt, "An Introduction to the Operations with Series," Chelsea Pub. Co., p. 177) that

\[
\cos \frac{\pi}{5} = \frac{1 + \sqrt{5}}{4}
\]

\[
\cos \frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}
\]
Therefore, 

\[ a = \frac{1 + \sqrt{5}}{2} = 2 \cos \frac{\pi}{5}, \]

\[ b = \frac{1 - \sqrt{5}}{2} = -2 \sin \frac{\pi}{10} \]

and

\[
F_n = \frac{a^n - b^n}{a - b} = 2^{n-1} \left[ \cos \frac{n \pi}{5} - (-1)^n \sin \frac{n \pi}{10} \right]
\]

\[
= 2^{n-1} \frac{\cos \frac{n \pi}{5} - \sin \frac{n \pi}{10}}{\cos \frac{\pi}{5} + \sin \frac{\pi}{10}} = 2^{n-1} \sum_{k=0}^{n-1} \cos \frac{n-k-1}{5} \pi \sin \frac{k}{10}
\]

as stated. We have made use of the algebraic identity

\[
\frac{x^n - y^n}{x - y} = \sum_{k=0}^{n-1} x^{n-k-1} y^k.
\]

Also solved by Charles R. Wall, Texas Christian University, who pointed out that the identity does not hold for \( n = 0 \).

A TELESCOPING SUM

B-19 Proposed by L. Carlitz, Duke University, Durham, N.C.

Show that

\[
\sum_{n=1}^{\infty} \frac{1}{F_n F_{n+2}} + \sum_{n=1}^{\infty} \frac{1}{F_n F_{n+1} F_{n+3}} = \frac{1}{2}
\]

Solution by John H. Avila, University of Maryland, College, Park Maryland

Our solution is similar to that by Francis D. Parker for B-9. Let \( a = a(n) = F_n \), \( b = F_{n+1} \), \( c = F_{n+2} \), and \( d = F_{n+3} \). Then \( a + b = c \), \( b + c = d \), and the left side of the desired formula is
\[
\sum_{n=1}^{\infty} \left( \frac{1}{ac^2d} + \frac{1}{ab^2d} \right) = \sum_{n=1}^{\infty} \left( \frac{b}{abc^2d} + \frac{c}{ab^2cd} \right)
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{c - a}{abc^2d} + \frac{d - b}{ab^2cd} \right)
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{abcd} - \frac{1}{bc^2d} + \frac{1}{ab^2c} - \frac{1}{abcd} \right)
\]

\[
= \sum_{n=1}^{\infty} \left( \frac{1}{ab^2c} - \frac{1}{bc^2d} \right).
\]

The last sum is the telescoping series

\[
\left( \frac{1}{F_1 F_2 F_3} - \frac{1}{F_2 F_3 F_4} \right) + \left( \frac{1}{F_2 F_3 F_4} - \frac{1}{F_2 F_3 F_5} \right) + \cdots
\]

whose sum is

\[
\frac{1}{F_1 F_2^2 F_3} = \frac{1}{1 \cdot 1^2 \cdot 2} = \frac{1}{2}.
\]

Also solved by the proposer.

**SUMMING GENERALIZED FIBONACCI NUMBERS**

B-20  Proposed by Louis G. Brökling, Redwood City, California

Generalize the well-known identities,

(i) \(F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1\)

(ii) \(L_1 + L_2 + L_3 + \cdots + L_n = L_{n+2} - 3\).
Solution by Charles R. Wall, Texas Christian University, Ft. Worth, Texas

If \( H_0 = q, \ H_1 = p, \) and \( H_{n+2} = H_{n+1} + H_n, \) then \( H_n = pF_n + qF_{n-1} \)
so that

\[
\sum_{i=1}^{n} H_i = p \sum_{i=1}^{n} F_i + q \sum_{i=0}^{n-1} F_i = p(F_{n+2} - 1) + q(F_{n+1} - 1)
\]

\[
= pF_{n+2} + qF_{n+1} - (p+q) = H_{n+2} - (p+q) = H_{n+2} - H_2.
\]


Also solved by Fern Grayson, Lockheed Missiles and Space Company, Sunnyvale California and the proposer.

**Evens and Odds**

**B-21** Proposed by L. Carlitz, Duke University, Durham, N. C.

If

\[
u_n = \frac{1}{2} [(x+1)^n + (x-1)^n]
\]

show that

\[
u_{n+1} = u_n^2 + 2^{2n} u_n^2 u_1^2 \cdots u_{n-1}^2
\]

Solution by Robert Means, University of Michigan

Let \( v_n = (x+1)^n - u_n. \) Then \( u_0 = x, \ v_0 = 1 \) and for \( n \geq 1, u_n \) and \( v_n \) are the terms of even and of odd degree respectively in \( (x+1)^2. \) Now

\[
u_{n+1} + v_{n+1} = (u_n + v_n)^2 = u_n^2 + 2u_n v_n + v_n^2
\]

and equating sums of terms of even and of odd degree respectively we have for \( n \geq 0,

(a) \quad u_{n+1} = u_n^2 + v_n^2
(b) \quad v_{n+1} = 2u_n v_n
Repeated use of (b) leads to 
\[ n = 2u_{n-1}v_{n-1} = 2u_{n-1}u_{n-2}v_{n-2} = \cdots = 2^n u_{n-1}u_{n-2} \cdots u_0 v_0. \]
Since \( v_0 = 1 \), the desired result is obtained by substituting the last expression for \( v_n \) in (a).

*Also solved by Charles R. Wall, Texas Christian University and the proposer*

**LUCAS ANALOGUES**

**B-22** Proposed by Brother U. Alfred, St. Mary's College, California

Prove the Fibonacci identity

\[ F_{2k} F_{2j} = F_{k+k'}^2 - F_{k-k'}^2 \]

and find the analogous Lucas identity.

(Editor's Note: The Fibonacci identity here is proved by I. D. Ruggles in "Some Fibonacci Results Using Fibonacci-Type Sequences," this Quarterly, Vol. 1, Issue 2, p. 77.) Proofs were submitted by Douglas Lind, Falls Church, Virginia; V. E. Hoggatt, Jr., San Jose State College; and Charles R. Wall, Texas Christian University, Ft. Worth, Texas. Lind and Hoggatt gave

\[ L_{2k} L_{2j} = L_{k+j}^2 + L_{k-j}^2 - 4(-1)^{k-j} \]

as the analogous Lucas identity and Wall gave it as

\[ L_{2k} L_{2j} = L_{k+j}^2 + 5F_{k-j}^2 = 5F_{k+j}^2 + L_{k-j}^2. \]

Proofs of these are left to the readers.

**TELESCOPING PRODUCTS AND SUMS**

**B-23** Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, Calif.

Prove the identities

\[(i)\]

\[ F_{n+1} = \prod_{i=1}^{n} \left( 1 + \frac{F_{i-1}}{F_i} \right) \]
### ELEMENTARY PROBLEMS AND SOLUTIONS

#### 1964

#### Solution by J. L. Brown, Jr., Pennsylvania State University, State College, Pa.

(i) \[
F_{n+1} = \frac{F_{n+1}}{F_n} \cdot \frac{F_{n-1}}{F_{n-2}} \cdot \frac{F_2}{F_1} = \prod_{i=1}^{n} \frac{F_{i+1}}{F_i} = \prod_{i=1}^{n} \frac{F_1 + F_2}{F_1} = \prod_{i=1}^{n} \left(1 + \frac{F_{i-1}}{F_1}\right).
\]

(ii) \[
\frac{F_{n+1}}{F_n} = \left(\frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}}\right) + \left(\frac{F_n}{F_{n-1}} - \frac{F_{n-1}}{F_{n-2}}\right) + \cdots + \left(\frac{F_3}{F_2} - \frac{F_2}{F_1}\right) + 1
\]

\[
= 1 + \sum_{i=2}^{n} \left(\frac{F_{i+1}}{F_i} - \frac{F_i}{F_{i-1}}\right) = 1 + \sum_{i=2}^{n} \frac{F_{i+1}F_{i-1} - F_i^2}{F_iF_{i-1}}
\]

\[
= 1 + \sum_{i=2}^{n} (-1)^i \frac{F_i}{F_{i-1}}
\]

using the well-known identity,

\[
F_{i+1}F_{i-1} - F_i^2 = (-1)^i.
\]

(iii) In (ii) take the limit as \( n \to \infty \) and recall that \( \lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2} \).

Also solved by Dermott A. Breault, Sylvania-ARL, Waltham, Mass.; Douglas Lind, Falls Church, Va; Charles R. Wall, Texas Christian University, Ft. Worth, Texas; and the proposer.

### A CORRECTED SOLUTION

#### B-4 Proposed by S. L. Basin, Sylvania Electronic Systems, Mt. View, California, and Vladimir Ivannoff, San Carlos, California

Show that \( \sum_{i=0}^{n} \binom{n}{i} F_i = F_{2n} \)

Generalize.
Using the Binet formula,

\[ F_{2n+j} = \frac{(a^j)^n b^j - (b^j)^n a^j}{a - b} = \frac{(1 + a)^n a^j - (1 + b)^n b^j}{a - b} \]

since

\[ a^2 = a + 1, \quad b^2 = b + 1 \quad \text{when} \quad a = \frac{1 + \sqrt{5}}{2}, \quad b = \frac{1 - \sqrt{5}}{2}, \]

we have

\[ F_{2n+j} = \frac{1}{a - b} \left[ \sum_{i=0}^{n} \binom{n}{i} a^{i+j} - \sum_{i=0}^{n} \binom{n}{i} b^{i+j} \right] = \sum_{i=0}^{n} \binom{n}{i} \frac{a^{i+j} - b^{i+j}}{a - b} = \]

\[ \sum_{i=0}^{n} \binom{n}{i} F_{i+j} \cdot \]

Therefore, for arbitrary integral \( j \),

\[ F_{2n+j} = \sum_{i=0}^{n} \binom{n}{i} F_{i+j} \cdot \]

If \( j = 0 \), we have the original problem. The identity also holds, with arbitrary \( j \), for Lucas numbers since \( L_n = F_{n+1} + F_{n-1} \).

CORRECTION TO VOLUME 1, NO. 1

See Vol. 1, No. 2, p. 46 for correction to last two references on page 42.