STRENGTHENED INEQUALITIES FOR FIBONACCI AND LUCAS NUMBERS

DOV JARDEN
Jerusalem, Israel

In a paper entitled "On the Greatest Primitive Divisors of Fibonacci and Lucas Numbers" (henceforth referred to as P), published in The Fibonacci Quarterly, Volume 1, Number 3, pages 15 – 20, I have proved for the Fibonacci numbers F_n and the Lucas numbers L_n the following inequalities:

(4) $$F_{n^{x+1}} > F^2_{n^x} \quad (n \geq 2,\ x \geq 1)$$

(5) $$F_{5^{x+1}} > 5F^2_{5^x} \quad (x \geq 1)$$

(4*) $$L_{n^{x+1}} > L^2_{n^x} \quad (n > 2,\ x \geq 1)$$

The aim of this note is to strengthen (4), (5), and (4*) as follows:

(A) $$F_{n^{x+1}} > nF^n_{n^x} \quad (n \geq 2,\ x \geq 1)$$

(B) $$L_{n^{x+1}} > L^{n-1}_{n^x} \quad (n \geq 2,\ x \geq 1)$$

For the proof of (A), (B) we shall use the well-known formulae

(C) $$F_n = \frac{1}{\sqrt{5}} \{\alpha^n - (-1)^n \alpha^{-n}\}$$

(D) $$L_n = \alpha^n + (-1)^n \alpha^{-n}$$

$$\alpha = \frac{1 + \sqrt{5}}{2} > \frac{3}{2}, \quad \alpha^{n+2} = \alpha^{n+1} + \alpha^n$$

as well as the following inequalities:

(E) $$\frac{1}{\sqrt{5}} \cdot \frac{6}{7} \cdot 2^n > n \quad (n \geq 3)$$

(F) $$\frac{1}{2} \alpha^n > F_n \quad (n \geq 2)$$

(G) $$\frac{6}{5} \alpha^n > L_n \quad (n \geq 2)$$

Proof of (E) (by induction). (E) is equivalent to

$$(E') \qquad 6 \cdot 2^n > 7n\sqrt{5}$$

(E') is valid for n = 3. If (E') is valid for n, then:

$$6 \cdot 2^{n+1} = 6 \cdot 2^n + 6 \cdot 2^n > 7n\sqrt{5} + 7n\sqrt{5} > 7n\sqrt{5} + 7\sqrt{5} = 7\sqrt{5}(n+1) .$$

Proof of (F), (G) (by induction on n and n + 1).

(F) is valid for n = 2, 3, since

$$\alpha^2 = 1 + \alpha = 1 + \frac{1+\sqrt{5}}{2} = \frac{3+\sqrt{5}}{2} > \frac{3+\sqrt{4}}{2} > 2 = 2F_2 ,$$

$$\alpha^3 = \alpha + \alpha^2 = \frac{1+\sqrt{5}}{2} + \frac{3+\sqrt{5}}{2} = 2 + \sqrt{5} > 2 + \sqrt{4} = 4 = 2F_3 .$$

If

$$\alpha^n > 2F_n ,$$
$$\alpha^{n+1} > 2F_{n+1} ,$$

then also:

$$\alpha^{n+2} = \alpha^n + \alpha^{n+1} > 2(F_n + F_{n+1}) = 2F_{n+2} .$$

(G) may be proven analogously, noting that, by arguments employed in the proof of (F), (G) is valid for n = 2, 3, since

$$\frac{6}{5}\alpha^2 > \frac{6}{5} \cdot \frac{3+\sqrt{4}}{2} = 3 = L_2 ,$$

$$\frac{6}{5}\alpha^3 > \frac{6}{5} \cdot 4 > 4 = L_3 .$$

Proof of (A).

(1) For n = 2 we have, by (C):

$$F_{2^{x+1}} = \frac{1}{\sqrt{5}}\{\alpha^{2^{x+1}} - \alpha^{-2^{x+1}}\} = \frac{\sqrt{5}}{5}\{\alpha^{2^{x+1}} - \alpha^{-2^{x+1}}\} > \frac{2}{5}\{\alpha^{2^{x+1}} - \alpha^{-2^{x+1}}\} >$$

$$\frac{2}{5}\{\alpha^{2^{x+1}} - (2 - \alpha^{-2^{x+1}})\} = \frac{2}{5}\{\alpha^{2^{x+1}} - 2 + \alpha^{-2^{x+1}}\} = 2\left\{\frac{1}{\sqrt{5}}(\alpha^{2^x} - \alpha^{-2^x})\right\}^2 = 2F^2_{2^x} .$$

(2) For $n \geq 3$ we have, by arguments employed in the proof of (F),

$$\alpha^{n^{x+1}} \geq \alpha^{3^2} = (\alpha^3)^3 > 4^3 > 7 ,$$

i.e.,

$$\frac{\alpha^{n^{x+1}}}{7} > 1 .$$

Hence, by (C), (E):

$$F_{n^{x+1}} = \frac{1}{\sqrt{5}}\{\alpha^{n^{x+1}} - (-1)^n \alpha^{-n^{x+1}}\} > \frac{1}{\sqrt{5}}\left\{\alpha^{n^{x+1}} - \frac{\alpha^{n^{x+1}}}{7}\right\} =$$

$$\frac{1}{\sqrt{5}} \cdot \frac{6}{7}\alpha^{n^{x+1}} = \frac{1}{\sqrt{5}} \cdot \frac{6}{7} \cdot 2^n \left(\frac{\alpha^{n^x}}{2}\right)^n > n F^n_{n^x} .$$

Proof of (B). For $n \geq 2$ we have $(n^x - 1)/(n - 1) = n^{x-1} + n^{x-2} + \cdots + 1 \geq n^{x-1} \geq (n - 1)^{x-1}$, whence: $n^x - 1 \geq (n - 1)^x$. Hence, by (D), (G), and noting that (by arguments employed in the proof of (A), part (2)) $-\alpha^{-n^{x+1}} > -\frac{1}{7}$ we have:

$$L_{n^{x+1}} = \alpha^{n^{x+1}} + (-1)^n \alpha^{-n^{x+1}} \geqslant \alpha^{n^{x+1}} - \alpha^{-n^{x+1}} > \alpha^{n^{x+1}} - \frac{1}{7} >$$

$$\alpha^{n^{x+1}} - \frac{1}{3}\alpha^{n^{x+1}} = \frac{2}{3}(\alpha^{n^x})^n > \frac{1}{\alpha}(\alpha^{n^x})^n = \alpha^{n^x - 1}(\alpha^{n^x})^{n-1} >$$

$$\alpha^{(n-1)^x}(\alpha^{n^x})^{n-1} \geq \alpha^{n-1}(\alpha^{n^x})^{n-1} > \left(\frac{6}{5}\right)^{n-1}(\alpha^{n^x})^{n-1} =$$

$$\left(\frac{6}{5}\alpha^{n^x}\right)^{n-1} > L^{n-1}_{n^x} .$$

Remark. In proving the inequalities (A), (B), I was assisted by my son, Moshe, who also noted that (B) cannot be strengthened, analogously to (A), to: $L_{n^{x+1}} > L^n_{n^x}$. Indeed, for $n = 4$, $x = 1$, we have: $L_{4^2} = 2207 < 2401 = 7^4 = L^4_4$.

It may also easily be seen, by (C), (D), that

(H) $$\lim_{x \to \infty} \frac{F_{n^{x+1}}}{n F^n_{n^x}} = \infty$$

(I) $$\lim_{x\to\infty} \frac{L_{n^{x+1}}}{L^{n-1}_{n^x}} = \infty$$

which shows that, for any given $n \geq 2$, there exists an X such that, for any $x > X$, $F_{n^{x+1}} > nF^n_{n^x}$, $L_{n^{x+1}} > L^{n-1}_{n^x}$.

By means of (A), (B), and employing the same reasoning as in the proof of (3), (3*) in P, we have, for the greatest primitive divisors F'_n of F_n and L'_n of L_n, the following generalized inequalities:

(J) $$F'_{p^{x+1}} > p\,F^{p-1}_{p^x} \quad (p\text{ — a prime} \neq 5,\ p \geq 2,\ x \geq 1)$$

(K) $$F'_{5^{x+1}} > F^4_{5^x} \quad (x \geq 1)$$

(L) $$L'_{p^{x+1}} > L^{p-2}_{p^x} \quad (p\text{ — a prime},\ p \geq 2,\ x \geq 1)\ .$$

SOME CORRECTIONS TO VOLUME 1, NO. 3

Page 16: In Equation (4*), replace $n \geq 2$ by $n > 2$.

The last line should read:

. . . for any positive integer $n \geq 2$, $n > 2$, respectively.

Page 17: On line 6, add $>$ to read:

$$\alpha = \frac{1 + \sqrt{5}}{2} > \frac{1 + \sqrt{4}}{2} = \frac{3}{2}$$

Line 8, Equation (7), should be corrected to read:

$$\alpha > \frac{3}{2}$$

On Line 11, add = to read:

$$\beta = \frac{1 - \sqrt{5}}{2} < \frac{1 - \sqrt{4}}{2} = -\frac{1}{2}$$