A FIBONACCI TEST FOR CONVERGENCE

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Let g(n) be a non-increasing positive function defined on the positive integers. There are many available tests to determine whether or not $\sum_{n=1}^{\infty} g(n)$ converges. It is the purpose of this paper to exhibit a test for convergence which utilizes the Fibonacci numbers.

THE FIBONACCI TEST

$\sum_{n=1}^{\infty} g(n)$ converges if and only if $\sum_{n=1}^{\infty} f_n g(f_n)$ converges, where f_n is the nth Fibonacci number.

Proof: Assume $\sum_{n=1}^{\infty} g(n)$ converges.

$$
\begin{array}{ll}
\frac{1}{2} g(1) & = \frac{1}{2} f_1 g(f_2) \\
\frac{1}{2} \{g(1) + g(2)\} & \geq \frac{1}{2} f_2 g(f_3) \\
\frac{1}{2} \{g(2) + g(3) + g(4) + g(5)\} & \geq \frac{1}{2} f_3 g(f_4) \\
\frac{1}{2} \{g(3) + g(4) + g(5) + g(6) + g(7) + g(8)\} & \geq \frac{1}{2} f_4 g(f_5) \\
\vdots & \\
\frac{1}{2} \{g(f_{n-2} + 1) + \cdots + (g(f_n))\} & \geq \frac{1}{2} f_{n-1} g(f_n) \\
\vdots &
\end{array}
$$

The sum of all terms on the left side of this array is $\sum_{n=1}^{\infty} g(n)$. The sum of all terms on the right side of the array is

$$\frac{1}{2} \sum_{n=2}^{\infty} f_{n-1} g(f_n) \geq \frac{1}{4} \sum_{n=2}^{\infty} f_n g(f_n) \quad .$$

Since the left side dominates the right side $\sum_{n=1}^{\infty} f_n g(f_n)$ converges.

Assume that $\sum_{n=1}^{\infty} g(n)$ diverges.

$$
\begin{aligned}
f_2 g(f_1) &= g(1) \\
f_3 g(f_2) &\geq g(1) + g(2) \\
f_4 g(f_3) &\geq g(2) + g(3) + g(4) \\
&\;\vdots \\
f_{n+1} g(f_n) &\geq g(f_n) + \cdots + g(f_{n+2} - 1) \\
&\;\vdots
\end{aligned}
$$

The sum of all terms on the right side of this array is $2 \sum_{n=1}^{\infty} g(n)$. The sum of all terms on the left side of this array is $\sum_{n=1}^{\infty} f_{n+1} g(f_n) \leq 2 \sum_{n=1}^{\infty} f_n g(f_n)$. Since $\sum_{n=1}^{\infty} g(n)$ diverges so does $\sum_{n=1}^{\infty} f_n g(f_n)$.

FURTHER REMARKS

It should be noticed that this result can be generalized to the following:

Theorem: If $1 \leq c_k = [H \cdot a^k]^*$ where $a > 1$ and H is a fixed positive constant then $\sum_{n=1}^{\infty} g(n)$ converges if and only if $\sum_{k=1}^{\infty} c_k g(c_k)$ converges.

The proof is quite similar.

It seems unlikely that this Fibonacci test for convergence will ever become widely used. To designate some of its useful qualities the following results are exhibited.

Corollary 1: $\sum_{n=1}^{\infty} n^a$ converges or diverges as $\sum_{n=2}^{\infty} n^{-1} \ln^a n$ does, $a < 0$.

Proof: Let r be the golden ratio, i.e., $r = (1 + \sqrt{5})/2$ and notice that $f_{n-1} = [r^n/\sqrt{5}]^*$. Now $\sum_{n=1}^{\infty} n^a$ converges or diverges as $\sum_{n=1}^{\infty} n^a \ln^a r$ does. But $n^a \ln^a r = (\ln r^n)a$ which is approximately $(\ln \sqrt{5} f_{n-1})^a$. So $\sum_{n=1}^{\infty} n^a \ln^a r$ converges or diverges as $\sum_{n=1}^{\infty} (\ln \sqrt{5} f_n)^a$ does. Now

$$\sum_{n=1}^{\infty} (\ln \sqrt{5} f_n)^a = \sum_{n=1}^{\infty} f_n \cdot f_n^{-1} (\ln \sqrt{5} f_n)^a$$

and appealing to the Fibonacci test this converges or diverges as

*[x] is greatest integer in x.

$$\sum_{n=1}^{\infty} n^{-1} (\ln \sqrt{5}\, n)^a$$

does, which is essentially the desired result.

This corollary tells us for example that since the harmonic series diverges then $\sum_{n=2}^{\infty} n^{-1} \ln n$ diverges and since $\sum_{n=1}^{\infty} n^{-1-\epsilon}$ converges then $\sum_{n=2}^{\infty} n^{-1}(\ln n)^{-1-\epsilon}$ converges.

<u>Corollary 2:</u> $\sum_{n=2}^{\infty} n^{-1} \ln^a n$ converges or diverges as

$$\sum_{n=3}^{\infty} (n \ln n)^{-1} (\ln \ln n)^a$$

does.

The proof is quite similar.

<u>Corollary j:</u> $j = 3, 4, 5, \cdots$ are likewise provable.

The Fibonacci test is an effective substitute for the integral test in each of these corollaries.

Consider the following example that is handled easily by the Fibonacci test. Let $g(n) = f_m^a$ for $f_{m-1} < n \le f_m$, where $a < 0$. Thus

$$\sum_{n=1}^{\infty} g(n) = 1^a + 2^a + 3^a + 5^a + 5^a + 8^a + 8^a + 8^a + 13^a + \cdots$$

Applying the Fibonacci test one obtains

$$\sum_{n=1}^{\infty} f_n g(f_n) = \sum_{n=1}^{\infty} f_n^{a+1} = \sum_{n=1}^{\infty} \left[\frac{r^{n+1}}{\sqrt{5}}\right]^{(a+1)} ,$$

which converges or diverges as $\sum_{n=1}^{\infty} (r^{n+1})^{a+1} = \sum_{n=1}^{\infty} (r^{a+1})^{n+1}$, but this is a geometric progression and converges provided $r^{a+1} < 1$ or when $a < -1$.