TRANSCENDENTAL NUMBERS
BASED ON THE FIBONACCI SEQUENCE

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A well-known theorem due to Liouville states that if $\xi$ is an irrational algebraic number of degree $n$, then the equation

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^{n+\epsilon}}$$

has only finitely many solutions for integers $p, q$, given any $\epsilon > 0$. Therefore, an irrational number $\xi$, for which

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^t}$$

has solutions for arbitrarily large $t$, must be transcendental. Numbers of this type have been called Liouville numbers.

In 1955, Roth published his celebrated improvement of Liouville's theorem, replacing "n" by "2" in equation (1). Let us call an irrational number $\xi$, for which

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{2^{n+\epsilon}}$$

has infinitely many solutions for some $\epsilon > 0$, a Roth number. Roth numbers are also transcendental, and they include many more numbers than the Liouville numbers.

Let $b$ be an integer greater than 1. Then we define $\xi_b$ to be the continued fraction

$$\xi_b = \frac{1}{bF_0} + \frac{1}{bF_1} + \frac{1}{bF_2} + \ldots$$

Theorem: $\xi_b$ is a Roth number, hence $\xi_b$ is transcendental.
Proof: From the elementary theory of continued fractions, it is well known that if \( \frac{p_n}{q_n} \) is the \( n \)th convergent to \( \xi_b \), then

\[
|\xi_b - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}.
\]

In this case, \( q_0 = 1 \), \( q_1 = b^F \), and \( q_{n+1} = b^F q_n + q_{n-1} \). We can therefore easily verify by induction that

\[
q_n = \frac{b^{F_{n+1}} - 1}{b - 1}.
\]

In particular, as \( n \to \infty \) we have \( \frac{q_{n+1}}{q_n} \to b^{F_{n+2}} / b^{F_{n+1}} = b^F \approx (b - 1)q_n^\Phi \) where \( \phi = .618 \cdots \) is the golden ratio. Therefore for large \( n \) we have approximately

\[
|\xi_b - \frac{p_n}{q_n}| < \frac{1}{q_n^{2+\phi}}
\]

and this completes the proof of the theorem.

Remarks. It can be easily shown that the set of Roth numbers is of measure zero, but it is uncountable. For example, the number \( \sum_{n=1}^{\infty} b^{-c_n} \), where \( \{c_n\} \) is a strictly increasing sequence of positive integers, is a Roth number if \( \lim \sup_{n \to \infty} (c_{n+1}/c_n) > 2 \), and it is a Liouville number if this \( \lim \sup \) is infinite. In terms of continued fractions, the number

\[
\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots
\]

is a Roth number if and only if

\[
\lim \sup_{n \to \infty} \left( \log \frac{a_n}{\log q_n} \right) > 0
\]

where \( q_n \) are the denominators as in the proof of the above theorem.

The rapid convergence of (4) allows us to evaluate \( \xi_b \) easily with high precision, e.g.,

\[
\xi_2 = .70980 34448 61291 \cdots
\]

\[
\xi_3 = .76859 75625 93155 \cdots
\]

Reference to this article on p. 52.