A NEW PROOF FOR AN OLD PROPERTY*

GLEN MICHAEL
Washington State University, Pullman, Wash.

1. INTRODUCTION

The following theorem is certainly well known.

Theorem: If m and n are positive integers, then $(F_m, F_n) = F_{(m,n)}$. For example, proofs can be found in [1, pp. 30–32] and [2, pp. 148–149]. In this paper we give an alternative proof which is believed to be new.

2. PRELIMINARY RESULTS

In addition to elementary divisibility properties of integers, the proof depends on the following lemmas which may be found in [1, pp. 10, 30 and 29].

Lemma 1: For $n \ge 0$,

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+1} .$$

Lemma 2: For any n, $(F_n, F_{n+1}) = 1$.

Lemma 3: For $n \neq 0$, $F_n \mid F_{mn}$.

3. PROOF OF THE THEOREM

For $m \ge 1$, $n \ge 1$, we show that $(F_m, F_n) = F_{(m,n)}$. Let

$$c = (m, n)$$

Then $c \mid m$, $c \mid n$ and, by Lemma 3, $F_c \mid F_m$ and $F_c \mid F_n$. Thus, F_c is a common divisor of F_m and F_n and it follows that $F_c \mid d$ where $d = (F_m, F_n)$. Also, since $c = (m,n)$, there exist integers a and b such that

$$c = am + bn$$

*This paper stems from a talk prepared under the guidance of Professor C. T. Long for presentation to the Washington State University Mathematics Club.

Since $c \leq m$ and m and n are positive, either $a \leq 0$ or $b \leq 0$. Suppose $a \leq 0$ and set $k = -a$. Then

$$bn = c + km$$

and, by Lemma 1,

$$F_{bn} = F_{c+km} = F_{c-1} F_{km} + F_c F_{km+1} . \qquad (1)$$

Now $d \mid F_n$, $d \mid F_m$ and, by Lemma 3, $F_n \mid F_{bn}$ and $F_m \mid F_{km}$. Therefore, $d \mid F_{bn}$, $d \mid F_{km}$ and it follows from (1) that $d \mid F_{km+1} F_c$. But $(d, F_{km+1}) = 1$ since $d \mid F_{km}$ and by Lemma 2, $(F_{km}, F_{km+1}) = 1$. Therefore, $d \mid F_c$. But, as seen above, $F_c \mid d$. Hence, since both are positive,

$$(F_m, F_n) = d = F_c = F_{(m,n)}$$

and the proof is complete.

REFERENCES

1. N. N. Vorob'ev, Fibonacci Numbers, Blaisdell Publishing Company, New York and London, 1961.
2. G. H. Hardy and E. M. Wright, The Theory of Numbers, Oxford University Press, London, 1954.

SOME CORRECTIONS TO VOLUME 1, NO. 3

Page 19: On the third line from the bottom, put in > for = to read

$$(5 + \beta^{n^{x+1}}) > .$$

Page 24: Line 5 should read, instead of "$a\alpha + 2\beta = 0$,"

$$a\alpha + 2b = 0 .$$

Page 30: On line 4, change "e_i" to "e_1" .
On line 18, change "unit" to "limit."