PARTITION ENUMERATION BY MEANS OF SIMPLER PARTITIONS

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1. INTRODUCTION

Netto [1] illustrates a method for enumerating all partitions of n having exactly p members, all non-zero. It is shown herein that Netto's procedure can be reduced to an algorithmic form through use of simpler partitions which are limited in range (size and number of members) but otherwise unrestricted. Properties of these range-limited partitions *per se* and a means of adapting them to Netto's enumeration procedure are discussed. An obvious application of the algorithmic procedure is the digital computation of both types of partitions.

2. LIMITED-RANGE, UNRESTRICTED PARTITIONS

Chrystal's [2] partition terminology, suitably modified, is used throughout. Thus, $P(\geqq n_1, \leqq n_2 | \geqq p_1, \leqq p_2 | \geqq q_1, \leqq q_2)$ specifies the enumeration of partitions of all positive integers from n_1 to n_2, inclusive, no partition having less than p_1 nor more than p_2 members, each member being not less than q_1 nor more than q_2. However, the *set* rather than the enumeration of range-limited partitions is of immediate interest in this paper, and to specify a set a V is appended to the enumeration notation. Accordingly, $PV(\geqq n_1, \leqq n_2 | \geqq p_1, \leqq p_2 | \geqq q_1, \leqq q_2)$ denotes the set of partitions having the properties of the enumeration counterpart.

The existence conditions are $n_2 \geqq n_1$, $p_2 \geqq p_1$, $q_2 \geqq q_1$, $q_2 p_2 \geqq n_1$, $q_1 p_1 \leqq n_2$, simultaneously. Moreover, for fixed n_1, n_2, q_1, q_2, there are optimum extreme values for p_1 and p_2. These are*

*Brackets [ ] except where obvious for references are used in the customary manner with real numbers to indicate the greatest integer less than or equal to the number bracketed. See Uspensky and Heaslet [3].

(1) $$p_{1\ opt.} = -[-(n_1/q_2)] \ ,$$

(2) $$p_{2\ opt.} = [n_2/q_1] \ .$$

If $p_1 \leqq p_{1\ opt.}$, p_1 can be changed to $p_{1\ opt.}$, but if $p_1 > p_{1\ opt.}$, p_1 cannot be changed. However, if $p_2 \geqq p_{2\ opt.}$, p_2 can be changed to $p_{2\ opt.}$, but if p_2 $p_{2\ opt.}$, p_2 cannot be changed.

In generating the partitions, the p_1-member partitions are found first, then the $(p_1 + 1)$-member partitions, etc., until the p_2-member partitions are found. The procedure used herein for the partitions of a typical p-member set is as follows:

A trial "first" partition is formed from p q_1's. If the sum of the p members is equal to or greater than n_1 but less than or equal to n_2, the partition initiates the set. If such is not so, the right-hand member is augmented so that the sum of the p-members in n_1. To form new partitions, the right-hand member is successively increased by one until either it equals q_2 or the sum of the p members equals n_2 (or both). The next p-member trial partition is found by adding one to the member second from the right and replacing all members to the right with the new value of the changed member. The desired reinitiating partition is found from the sum of the p members, as before. The right-hand member is successively increased by one to form new partitions. When the possibilities of the particular second member from the right are exhausted, one is added to the third member from the right and the process repeated all over again. Eventually, all p-member partitions will be accounted for. An example for PV(≥8, ≤10 |≥2, ≤5 |≥2, ≤7) follows:

2, 6	2, 2, 4	2, 2, 2, 2	2, 2, 2, 2, 2
2, 7	2, 2, 5	2, 2, 2, 3	
3, 5	2, 2, 6	2, 2, 2, 4	
3, 6	2, 3, 3	2, 2, 3, 3	
3, 7	2, 3, 4		
4, 4	2, 3, 5		
4, 5	2, 4, 4		
4, 6	3, 3, 3		
5, 5	3, 3, 4		

3. APPLICATION OF NETTO'S METHOD

Netto [1] considers the enumeration $P(n|p|\leq q)$ of the partitions of n having exactly p members with no member greater than q. Netto's method is limited to $q \geq (n+1-p)$ with the existence conditions being $p \leq n$ and $qp \geq n$, simultaneously. In the terminology of this paper,

$$(3) \qquad P(n|p|\leq q) = \sum_t \left[\tfrac{1}{2}(n-p+2-3t_1-4t_2- \ldots -pt_{p-2})\right] ,$$

where $t_\alpha = 0, 1, \ldots, \left[\frac{n-p+2}{\alpha+2}\right]$. Inspection of (3) reveals that the typical term is

$$(4) \qquad \left[\frac{n-p+2-w}{2}\right] ,$$

in which w is always zero for $t_\alpha = 0$, always 3 for $t_\alpha = 1$, and always greater than 3 for all other t_α's. It can be observed that except for the zero value of w, each w in the enumeration $P(n|p|\leq q)$ is the sum of the members of each partition included in the set

$$(5) \qquad PV(\geq 3, \leq n-p\,|\geq 1, \leq [\tfrac{n-p}{3}]\,|\geq 3, \leq p) .$$

Thus, except for $\rho = 1$,

$$(6) \qquad P(n|p|\leq q) = \left[\frac{n-p+2}{2}\right] + \sum_i \left[\frac{n-p+2-w_i}{2}\right] .$$

It should be noted that (5) does not exist for $p = 2$, and/or $(n-p) < 3$. There are no w_i's under these conditions, and the summation term of (6) is accordingly zero. The special case of $p = 1$ is

$$(7) \qquad P(n|1|\leq q) = 1 .$$

As was stated earlier, the methods described herein are particularly adaptable to digital computations. To this end, the author can supply a limited number of ALGOL language programs and test examples for enumerating partitions with the Burroughs 220 digital computer.

REFERENCES

1. E. Netto, Lehrbuch der Combinatorik, Leipsiz, 1901, pp. 127, 128.
2. G. Chrystal, Textbook of Algebra, Vol. 2, (Reprint) Chelsea Publishing Co., New York, 1952.
3. J. V. Uspensky and M. A. Heaslet, Elementary Number Theory, McGraw-Hill Book Co., New York, 1939, pp. 94-99.

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CORRECTIONS FOR VOLUME 1, NO. 3

Page 44: On line 4 read "$0 \leq k \leq 2^r - 1$" for "$0 \leq k \leq 2^r$"

Page 49: On line 8 read $[mF_n]/F_m$ for $[mF_n] F_m$

Page 80: In B-7 line 2 $x = 1/4$ and $\sum_{i=0}^{\infty} F_i^2/4^i = \frac{12}{25}$?

FURTHER CORRECTIONS FOR VOLUME 1, NO. 4

Reference 4 The first author is IVAN NIVEN .

In H-25 $(i, j = 1, 2, 3, 4)$