ON FIBONACCI RESIDUES

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In a recent note ("Exploring Fibonacci Residues" Fib. Quart. 2 (1964) 1: 42), Brother Alfred asks whether one or other of the least positive and negative residues, when one Fibonacci number is divided by another, is always itself a Fibonacci number.

The answer is YES, as is shown by the following somewhat more detailed result.

THEOREM. If $m \geq 1$ and $n \geq 3$ are integers, and if A and - B are the least positive and negative residues when F_m is divided by F_n, then at least one of A and B is itself a Fibonacci number F_s, where k and s are unique integers such that $s = 0$ if n divides m, and otherwise

(1) $$m = 2kn + r, \quad k \geq 0, \quad 0 < |r| < n, \quad s = |r| .$$

Proof. It is well-known that F_m is divisible by F_n if and only if either m is divisible by n or $n = 2$. Thus if $n \geq 3$ and n divides m, the theorem holds, since $F_0 = 0$. If n does not divide m, we can find k and r uniquely by (1). Well-known identities now show that

(2) $$F_m = F_{2kn+r} = \sum_{h=0}^{2k} \binom{2k}{h} F_n^h F_{n-1}^{2k-h} F_{r+h} \equiv F_{n-1}^{2k} F_r \pmod{F_n}$$

and

(3) $$F_{n-1}^2 = F_{n-2} F_n + (-1)^n \equiv (-1)^n \pmod{F_n} .$$

Therefore we see that

(4) $$F_m \equiv (-1)^{kn} F_r \equiv (-1)^{kn+r-1} F_{-r} \equiv \pm F_s \pmod{F_n} .$$

Since the Fibonacci sequence is strictly increasing for values of the index greater than one, $F_s < F_n$, so that $\pm F_s$ is the least positive or negative residue of F_m modulo F_n; that is, F_s = A or B.

To complete the treatment of Brother Alfred's question, it must be noted that, if n = 1 or 2, $F_n = 1$ and so divides F_m, yielding a residue of $F_0 = 0$. And if m or n is negative, the well-known relation

$$F_{-t} = (-1)^{t-1} F_t ,$$

which was used in the derivation of (4), shows that the residue is still $\pm F_s$.

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CORRECTED FACTORIZATIONS OF FIBONACCI NUMBERS

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Kraitchik's table of factors of the Fibonacci numbers (Recherches sur la Theorie des Nombres, " p. 77-79) contains at least two errors, as follows:

(u_n denotes n^{th} Fibonacci number, as in Kraitchik)

n	u_n	Kraitchik's Factorization	Correct Factorization
57	365, 435, 296, 162	2·37·113·4371901	2·37·113·797·54833
67	44, 945, 570, 212, 853	prime	269·116849·1429913

Note: in the factorization of u_{57}, 797 · 54833 = 43701901, not 4371901)
Have these errors been pointed out elsewhere?