FIBONACCI NUMBERS FROM A DIFFERENTIAL EQUATION

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In a course in differential equations, solving

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - y = 0 \quad (y = 0;\ y' = 1,\ x = 0)$$

leads to

(1) $$y = \frac{e^{\alpha x} - e^{\beta x}}{\alpha - \beta} = \sum_{n=0}^{\infty} \frac{\alpha^n - \beta^n}{\alpha - \beta} \frac{x^n}{n!} ,$$

where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$

both satisfy the auxiliary equation $m^2 - m - 1 = 0$.

On the other hand, solving this same problem directly in infinite series of the form

(2) $$y = \sum_{n=0}^{\infty} a_n x^n$$

leads to the recurrence relation

$$(n+2)(n+1)a_{n+2} - (n+1)a_{n+1} - a_n = 0 ,$$

with $a_0 = 0$, $a_1 = 1$.

If we set $a_n \equiv u_n/n!$ this becomes

$$(n+2)(n+1)\frac{u_{n+2}}{(n+2)!} - \frac{(n+2)\, u_{n+1}}{(n+1)!} - \frac{u_n}{n!} = 0$$

or

$$u_{n+2} - u_{n+1} - u_n = 0 ,$$

with $u_0 = 0$ and $u_1 = 1$.

Thus $$a_n = \frac{u_n}{n!} = F_n/n! ,$$

where F_n is the nth Fibonacci number.

Substituting these values of $u_n/n! = a_n$ into (2) yields

(3) $$y = \sum_{n=0}^{\infty} \frac{F_n}{n!} x^n$$

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