FOURTH POWER FIBONACCI IDENTITIES FROM PASCAL'S TRIANGLE

VERNER E. HOGGATT, JR. and MARJORIE BICKNELL
San Jose State College, San Jose, California

In this paper, matrix methods are used to derive some new fourth power Fibonacci identities. We let S be the 5×5 matrix which contains the first five rows of Pascal's triangle beneath and on its secondary diagonal; that is,

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} .$$

The right column elements of $S^n = (s_{ij})$ are given by

$$s_{i5} = \binom{4}{i-1} F_n^{5-i} F_{n+1}^{i-1} , \quad i = 1, 2, \ldots, 5.$$

Proof is by induction. Obviously, S has this form. Since

$$S^{n+1} = (t_{ij}) = SS^n ,$$

by definition of matrix multiplication,

$$t_{15} = F_{n+1}^4 ,$$

$$t_{25} = 4F_{n+1}^4 + 4F_n F_{n+1}^3 = 4F_{n+1}^3 F_{n+2} ,$$

$$t_{35} = 6F_{n+1}^4 + 12F_n F_{n+1}^3 + 6F_n^2 F_{n+1}^2 = 6F_{n+1}^2 (F_{n+1}^2 + 2F_n F_{n+1} + F_n^2)$$

$$= 6F_{n+1}^2 F_{n+2}^2 ,$$

$$t_{45} = 4F_{n+1}^4 + 12F_n F_{n+1}^3 + 12F_n^2 F_{n+1}^2 + 4F_n^3 F_{n+1}$$

$$= 4F_{n+1}(F_{n+1}^3 + 3F_{n+1}^2 F_n + 3F_{n+1} F_n^2 + F_n^3) = 4F_{n+1} F_{n+2}^3 ,$$

$$t_{55} = (F_n + F_{n+1})^4 = F_{n+2}^4 .$$

Since only the recursion relation of the Fibonacci sequence was used above, we have almost immediately a matrix identity for generalized Fibonacci numbers. Let u_n be the nth member of the generalized Fibonacci sequence defined by $u_1 = a$, $u_2 = b$, and $u_{n+1} = u_n + u_{n-1}$. Let $U = (a_{ij})$ be the column matrix defined by

$$a_{i1} = \binom{4}{i-1} u_1^{5-i} u_2^{i-1}, \quad i = 1, 2, \ldots, 5 .$$

By our earlier proof, we can write

$$S^n U = S^n \begin{bmatrix} u_1^4 \\ 4u_1^3 u_2 \\ 6u_1^2 u_2^2 \\ 4u_1 u_2^3 \\ u_2^4 \end{bmatrix} = \begin{bmatrix} u_{n+1}^4 \\ 4u_{n+1}^3 u_{n+2} \\ 6u_{n+1}^2 u_{n+2}^2 \\ 4u_{n+1} u_{n+2}^3 \\ u_{n+2}^4 \end{bmatrix} = U_{n+1} .$$

By the Cayley-Hamilton Theorem, S must satisfy the matrix equation

$$(1) \qquad S^n(S^5 - 5S^4 - 15S^3 + 15S^2 + 5S - I) = 0 .$$

Consideration of Equation (1) leads us to the matrix equation

$$(1') \qquad U_{n+5} - 5U_{n+4} - 15U_{n+3} + 15U_{n+2} + 5U_{n+1} - U_n = 0 ,$$

where U_n is defined as the matrix $S^{n-1}U$. Since the elements in the first rows of the matrices of Equation (1') must also satisfy the recursion relation of (1'), we have the identity

$$u_{n+5}^4 - u_n^4 = 5(u_{n+4}^4 + 3u_{n+3}^4 - 3u_{n+2}^4 - u_{n+1}^4) .$$

Equation (1) can be rewritten as

$$(S - I)^5 = 25S^2(S - I) .$$

It can easily be shown by induction that

$$(S - I)^{4n+1} = 25^n S^{2n}(S - I) .$$

We plan to investigate

$$(S - I)^{4n+1} = 25^n S^{2n}(S - I) , \tag{2}$$

$$(S - I)^{4n+2} = 25^n S^{2n}(S - I)^2 , \tag{3}$$

$$(S - I)^{4n+3} = 25^n S^{2n}(S - I)^3 , \tag{4}$$

$$(S - I)^{4n+4} = 25^n S^{2n}(S - I)^4 , \tag{5}$$

From Equation (2),

$$S^j(S - I)^{4n+1} = \sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} S^{i+j} = 25^n S^{2n+j}(S - I) .$$

Thus, equating elements in the upper right corner of these matrices,

$$\sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} F_{i+j}^4 = 25^n(F_{2n+j+1}^4 - F_{2n+j}^4) = A_j . \tag{2'}$$

Similarly, from Equations (3), (4), and (5), we obtain

$$\begin{aligned} \sum_{i=0}^{4n+2} (-1)^i \binom{4n+2}{i} F_{i+j}^4 &= 25^n(F_{2n+j+2}^4 - 2F_{2n+j+1}^4 + F_{2n+j}^4) \\ &= A_{j+1} - A_j , \end{aligned} \tag{3'}$$

$$\begin{aligned} \sum_{i=0}^{4n+3} (-1)^i \binom{4n+3}{i} F_{i+j}^4 &= 25^n(F_{2n+j+3}^4 - 3F_{2n+j+2}^4 + 3F_{2n+j+1}^4 - F_{2n+j}^4) \\ &= A_{j+2} - 2A_{j+1} + A_j , \end{aligned} \tag{4'}$$

$$\begin{aligned} \sum_{i=0}^{4n+4} (-1)^i \binom{4n+4}{i} F_{i+j}^4 &= 25^n(F_{2n+j+4}^4 - 4F_{2n+j+3}^4 + 6F_{2n+j+2}^4 \\ &\qquad - 4F_{2n+j+1}^4 + F_{2n+j}^4) \\ &= A_{j+3} - 3A_{j+2} + 3A_{j+1} - A_j . \end{aligned} \tag{5'}$$

But, also from (2),

$$\sum_{i=0}^{4n+5} (-1)^i \binom{4n+5}{i} F_{i+j}^4 = 25^{n+1}(F_{2n+j+3}^4 - F_{2n+j+2}^4) = 25\, A_{j+2} ,$$

so that we have the recursion relation

$$(6) \qquad A_{j+4} - 4A_{j+3} + 6A_{j+2} - 4A_{j+1} + A_j = 25\, A_{j+2} .$$

By use of well-known Fibonacci identities, we can rewrite Equations (2') and (4') respectively to yield the following:

$$\sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} F_{i+j}^4 = 25^n F_{2n+j-1} F_{2n+j+2} F_{2(2n+j)+1} ,$$

$$\sum_{i=0}^{4n+3} (-1)^i \binom{4n+3}{i} F_{i+j}^4 = 25^n L_{2n+j} L_{2n+j+3} F_{2(2n+j)+3} .$$

Equating elements in the first row of the column matrices formed by multiplying Equations (2) and (4) on the right by the matrix U, and taking $u_1 = 1$, $u_2 = 3$, we can rewrite Equations (2') and (4') to yield the identities

$$\sum_{i=0}^{4n+1} (-1)^i \binom{4n+1}{i} L_{i+j}^4 = 25^n(5 L_{2n+j-1} L_{2n+j+2} F_{2(2n+j)+1})$$

and

$$\sum_{i=0}^{4n+3} (-1)^i \binom{4n+3}{i} L_{i+j}^4 = 25^n(5 F_{2n+j} F_{2n+j+3} F_{2(2n+j)+3})$$

for fourth powers of members of the Lucas sequence $\{L_n\}$.

Returning to the recursion relation of Equation (6), we define

$$G(j) = F_{n+j+4}^4 - 4F_{n+j+3}^4 - 19F_{n+j+2}^4 - 4F_{n+j+1}^4 + F_{n+j}^4 .$$

By (6),

$$25^n(G(j+1) - G(j)) = A_{j+4} - 4A_{j+3} - 19A_{j+2} - 4A_{j+1} + A_j = 0 \quad .$$

Thus, $G(j+1) - G(j) = 0$, so that $G(j)$ is a constant. Taking $n = j = 0$, $G(j) = -6$, leading to an identity given by Zeitlin in [1]. If we redefine $G(j)$ by replacing the Fibonacci numbers by the corresponding Lucas numbers, we find that $G(j) = -150$. Further, if we replace members of the Fibonacci sequence by the corresponding generalized Fibonacci number, we obtain $G(j) = -6D^2$, where D is the characteristic of the sequence,

$$D = u_2^2 - u_1^2 - u_1u_2 \quad .$$

(See [2] and [3] for properties of the characteristic of Fibonacci-type sequences.)

Finally, we derive another property of the characteristic of a sequence. It is well-known that $F_{2n+3} = 3F_{2n+1} - F_{2n-1}$ and that $F_{2n+1} = F_{n+1}^2 + F_n^2$. Define

$$G(n) = F_{n+1}^2 - 3F_n^2 + F_{n-1}^2 \quad .$$

Then,

$$G(n+1) + G(n) = F_{2n+3} - 3F_{2n+1} + F_{2n-1} = 0 \quad ,$$

so that $G(n) = (-1)^n 2$. That is,

$$(-1)^n 2 = F_{n+1}^2 - 3F_n^2 + F_{n-1}^2 \quad .$$

For generalized Fibonacci numbers, it can be shown by induction that

$$R^nU = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}^n \begin{bmatrix} u_1^2 \\ 2u_1u_2 \\ u_2^2 \end{bmatrix} = \begin{bmatrix} u_{n+1}^2 \\ 2u_{n+1}u_{n+2} \\ u_{n+2}^2 \end{bmatrix} \quad .$$

The 3X3 matrix R given above has been discussed in an earlier article [4]. From the characteristic equation of R, we obtain

$$u_{n+3}^2 - 2u_{n+2}^2 - 2u_{n+1}^2 + u_n^2 = 0 .$$

Rewriting and defining H(n),

$$H(n) = u_{n+2}^2 - 3u_{n+1}^2 + u_n^2 = -u_{n+3}^2 - u_{n+1}^2 + 3u_{n+2}^2 = - H(n+1) .$$

Thus,

$$H(n) = (-1)^n(u_2^2 - 3u_1^2 + u_0^2) = (-1)^n 2(u_2^2 - u_1^2 - u_1 u_2) = (-1)^{n+1} 2D ,$$

where D is the characteristic of the sequence. That is,

$$(-1)^n 2D = u_{n+1}^2 - 3u_n^2 + u_{n-1}^2 .$$

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