GENERALIZED BINOMIAL COEFFICIENTS

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We consider the general second order recurrence relation (r. r.)

$$y_{n+2} = gy_{n+1} - hy_n, \quad h \neq 0 \quad . \tag{1}$$

Let a and b be the roots of the auxiliary polynomial $f(x) = x^2 - gx + h$ of (1). Using the notation of the classic paper [1] of E. Lucas, we let U_n and V_n be the solutions of (1) defined by $U_n = (a^n - b^n)/(a - b)$ if $a \neq b$ and $U_n = na^{n-1}$ if $a = b$ and by $V_n = a^n + b^n$.

In [3], D. Jarden defined generalized binomial coefficients by

$$\begin{bmatrix} m \\ j \end{bmatrix} = \frac{U_m U_{m-1} \cdots U_{m-j+1}}{U_1 U_2 \cdots U_j}, \quad \begin{bmatrix} m \\ 0 \end{bmatrix} = 1 \quad . \tag{2}$$

(We have changed Jarden's notation $\binom{m}{j}_U$ to $\begin{bmatrix} m \\ j \end{bmatrix}$.) If $g = 2$ and $h = 1$ then $U_n = n$ and $\begin{bmatrix} m \\ j \end{bmatrix}$ is the ordinary binomial coefficient $\binom{m}{j}$.

Jarden showed that the product z_n of the n-th terms of $k - 1$ sequences satisfying (1) satisfies the k-th order r. r.

$$\sum_{j=0}^{k} (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} h^{j(j-1)/2} z_{n+k-j} = 0. \tag{3}$$

The definition (2) of $\begin{bmatrix} m \\ j \end{bmatrix}$ for all j and m with $0 \leq j \leq m$ obviously requires that $U_n \neq 0$ for $n > 0$ since otherwise (2) may involve division by zero. We call the r. r. (1) <u>ordinary</u> if $U_n \neq 0$ for all $n > 0$ and <u>exceptional</u> if $U_n = 0$ for some $n > 0$. In (7) and (8) below we give an alternate definition of $\begin{bmatrix} m \\ i \end{bmatrix}$ which is valid in all cases. In [2], D. H. Lehmer considered the exceptional r. r. 's (1) for which $g = \sqrt{f}$ and for which f and h are relatively prime. Lehmer's paper is concerned with divisibility properties of the sequences U_n and V_n.

It follows from $h \neq 0$ that $a \neq 0$ and $b \neq 0$. It is then clear from the definition of U_n that (1) is exceptional if and only if $a \neq b$ and $a^p = b^p$ for some positive integer p. If (1) is exceptional, $a \neq b$ and so every solution of (1) is of the form $y_n = c_1 a^n + c_2 b^n$. Then

*This work was supported by the Undergraduate Research Participation Program of the National Science Foundation through G-21681. The authors express their gratitude to NSF and to Dr. A. P. Hillman, Dr. D. G. Mead, Mr. R. M. Grassl, and Mr. J. A. Erbacher for much valuable assistance.

$y_{n+p} = c_1 a^{n+p} + c_2 b^{n+p} = a^p(c_1 a^n + c_2 b^n) = a^p y_n$ for all n. Conversely, one easily sees that $y_{n+p} = a^p y_n$ for all n and all solutions y_n of (1) implies that (1) is exceptional.

We show below that the following four conditions are equivalent to each other and hence to (1) being ordinary:

(a) Either $a = b$ or $a^n \neq b^n$ for all $n > 0$.

(b) Any solution y_n of (1) with two different terms equal to zero is identically zero.

(c) For all $k \geq 2$ the r.r. (3) is the lowest order r.r. satisfied by all term by term products of $k - 1$ sequences satisfying (1).

(d) Every solution of (3) is of the form

$$(4) \quad z_n = c_1 U_n^{k-1} + c_2 U_n^{k-2} U_{n+1} + c_3 U_n^{k-3} U_{n+1}^2 + \ldots + c_k U_{n+1}^{k-1} ,$$

i.e., the sequences $U_n^{k-j} U_{n+1}^{j-1}$ for $j = 1, \ldots, k$ form a basis for the vector space of all solutions of (3).

We shall also establish some identities involving the $\left[\begin{matrix} m \\ j \end{matrix}\right]$, one of which is the addition formula:

$$(5) \quad \sum_{j=0}^{k} (-1)^j \left[\begin{matrix} k \\ j \end{matrix}\right] h^{(j+1)j/2} U_{a_1+k-j} U_{a_2+k-j} \cdots U_{a_k+k-j} y_{n+k-j} = U_1 \cdots U_k y_{n+a_1+\ldots+a_k+[k(k+1)/2]},$$

for y_n and U_n satisfying (1) and n and the a's any integers.

If $a \neq b$, every solution of (1) is of the form $y_n = c_1 a^n + c_2 b^n$ and the term-by-term product of $k - 1$ sequences satisfying (1) is given by

$$(6) \quad z_n = c_1 (a^{k-1})^n + c_2 (a^{k-2} b)^n + c_3 (a^{k-3} b^2)^n + \ldots + c_k (b^{k-1})^n .$$

We therefore let

$$(7) \qquad f_k(x) = (x - a^{k-1})(x - a^{k-2} b) \ldots (x - b^{k-1})$$

and define $\left[\begin{matrix} k \\ j \end{matrix}\right]$ so that

(8) $$f_k(x) = \sum_{j=0}^{k} (-1)^j \left[{k \atop j}\right] h^{j(j-1)/2} x^{k-j} .$$

The $\left[{k \atop j}\right]$ defined by (8) is a generalization of the $\left\{{k \atop j}\right\}$ of L. Carlitz [4] defined by

$$(1 - t)(1 - qt) \ldots (1 - q^{k-1}t) = \sum_{j=0}^{k} (-1)^j q^{j(j-1)/2} \left\{{k \atop j}\right\} t^j .$$

See especially formulas (6.3) through (6.16) of [4].)

Then $f_k(x)$ is the auxiliary polynomial for the r.r. (3). The lowest order r.r. satisfied by the z_n of (6) is (3) if and only if the numbers $a^{k-1}, a^{k-2}b, \ldots, b^{k-1}$ are distinct. Since $a \neq 0$ and $b \neq 0$, this is equivalent to $a^j \neq b^j$ for $j = 1, \ldots, k-1$. Hence condition (c) is equivalent to (a) for $a \neq b$.

If $a = b$, every solution of (1) is given by $y_n = (c_1 + c_2 n)a^n$, the term-by-term product of $k-1$ sequences satisfying (1) is of the form

(9) $$z_n = (c_1 + c_2 n + \ldots + c_k n^{k-1})(a^{k-1})^n ,$$

and (3) is the lowest order r.r. satisfied by all the z_n of form (9). Thus (c) and (a) are equivalent in this case too. It is also easily seen that $h = a^2$ and $\left[{m \atop j}\right] = \binom{m}{j} a^{j(m-j)}$ when $a = b$.

<u>Lemma.</u>

A solution y_n of (1) that is not identically zero has $y_n = 0$ for two different values of n if and only if $a \neq b$ and there is a positive integer p such that $a^p = b^p$.

Proof.

First let $a = b$. Then $y_n = (c_1 + c_2 n)a^n$. If $y_u = 0 = y_v$ with $u \neq v$, then $(c_1 + c_2 u)a^u = 0 = (c_1 + c_2 v)a^v$. Since $a \neq 0$, it follows that $c_1 + c_2 u = 0 = c_1 + c_2 v$, $c_2(u - v) = 0$, and so $c_2 = 0$. Then $c_1 = 0$ and $y_n = 0$ for all n.

Now let $a \neq b$. Then $y_n = c_1 a^n + c_2 b^n$. If $y_u = 0 = y_v$ with $u > v$, $c_1 a^u + c_2 b^u = 0 = c_1 a^v + c_2 b^v$, and there exists a non-trivial solution for the c's if and only if the determinant $a^u b^v - b^u a^v = 0$. This is equivalent to $a^{u-v} = b^{u-v}$.

This shows that (a) and (b) are equivalent.

Corollary.

If v_n and w_n are solutions of (1) and $v_n = w_n$ for two values of n, then $v_n = w_n$ for all n.

This follows from the lemma and the fact that $v_n - w_n$ is also a solution of (1).

We next consider condition (d). First let (1) be ordinary. Let z_n be the term-by-term product of $k - 1$ solutions of (1). If we can find constants $c_1, \ldots, c_k$ such that (4) holds for $n = 1, 2, \ldots, k$ then the r.r. (3), which is satisfied by the sequences $U_n^{k-j} U_{n+1}^{j-1}$ and z_n, will make (4) hold for all n. Such c's can be found if the k by k determinant D with $d_{ij} = U_i^{k-j} U_{i+1}^{j-1}$ is not zero. Since (1) is ordinary, each of $U_1, U_2, \ldots, U_k$ is not zero and we can factor U_i^{k-1} out of the elements of the i-th row of D thus obtaining the Vandermonde determinant E with $e_{ij} = (U_{i+1}/U_i)^{j-1}$. Then E, and hence D, is not zero if and only if the ratios U_{i+1}/U_i are distinct. It is easily seen that $U_{s+1}/U_s = U_{t+1}/U_t$ if and only if $a^{s-t} = b^{s-t}$. This shows that (a) implies (d).

If (1) is exceptional, $a^p = b^p$ for some $p > 0$ and so $U_{n+p+1}/U_{n+p} = U_{n+1}/U_n$. Then for $k > p$, the determinant D is zero since it has proportional rows. It follows that one of the sequences $U_n^{k-j} U_{n+1}^{j-1}$ is a linear combination of the others, first for $1 \leq n \leq k$ and then, using (3), for all n. This implies that there is a solution of (3) not of the form (4) and so (d) implies (a).

We now go back to (7) and note that $ab = h$. Therefore we can write

$$f_{k+2}(x) = \left[(x-a^{k+1})(x-b^{k+1})\right] \left[(x-a^k b)\ldots(x-ab^k)\right]$$

$$f_{k+2}(x) = \left[x^2-(a^{k+1}+b^{k+1}) + h^{k+1}\right]\left[(x-a^{k-1}h)(x-a^{k-2}bh)\ldots(x-b^{k-1}h)\right]$$

$$f_{k+2}(x) = h^k(x^2 - V_{k+1}x + h^{k+1})\, f_k(x/h) \quad , \tag{10}$$

where V_n is the general Lucas sequence $a^n + b^n$. Formula (10) implies the following:

(11) $$\left[\begin{matrix}k\\j\end{matrix}\right]h^{k-j} + \left[\begin{matrix}k\\j+1\end{matrix}\right]V_{k+1} + \left[\begin{matrix}k\\j+2\end{matrix}\right]h^{j+2} = \left[\begin{matrix}k+2\\j+2\end{matrix}\right] ,$$

(12) $$f_{2m} = \prod_{j=1}^{m} (x^2 - V_{2j-1}h^{m-j} + h^{2m-1}) ,$$

(13) $$f_{2m+1} = (x - h^m) \prod_{j=1}^{m} (x^2 - V_{2j}h^{m-j} + h^{2m}) .$$

We next prove identity (5) when (1) is ordinary by induction on k. When $k = 1$, (5) becomes

(14) $$U_{a+1}y_{n+1} - hU_a y_n = y_{n+a+1} .$$

We consider n to be a constant and let a be the running index. Then both sides of (14) satisfy (1) and they are equal to one another for $a = 0$ and $a = -1$ since $U_{-1} = -1/h$, $U_0 = 0$, and $U_1 = 1$. Hence (14) holds for all a (and all n) by the Corollary.

Now we assume that (5) holds for $k = m-1$ and show that this implies (5) for $k = m$. We consider $a_1, \ldots, a_{m-1}$ and n to be constants and let a_m be the running index. Both sides of (5) satisfy (1). When $a_m = 0$, (5) becomes U_m times the identity for $k = m-1$ with each a_j replaced by $1 + a_j$. When $a_m = -m$, (5) reduces to U_m times the identity for $k = m-1$ using the easily established fact that $U_{-n} = -U_n h^{-n}$. Hence (5) is true for two values of a_m and thus true for all values by the Corollary.

We now turn to identity (5) in the exceptional case. From symmetric function theory and the definitions (7) and (8), it follows that for fixed h the $\left[\begin{matrix}m\\j\end{matrix}\right]$ are polynomials in g. For fixed values of y_0 and y_1 and h, the two sides of (5) are then continuous functions of g. Thus (5) for complex numbers g_0 and h_0 that make (1) exceptional can be established by having g approach g_0 (while h is fixed at h_0) through values for which (1) is ordinary. A sufficient condition for (1) to be ordinary is that $|a| \neq |b|$. Any point (g_0, h_0) is a limit of points (g, h_0) satisfying this sufficient condition for (1) to be ordinary.

A purely algebraic proof of identity (5) in the exceptional case can also be given.

Finally we consider the $\begin{bmatrix} m \\ j \end{bmatrix}$ when (1) is exceptional and g and h are both real. Since $a^p = b^p$ for some $p > 0$, $|a| = |b|$. Since $a \neq b$ this means that $a = -b$, $g = 0$, and $h = -a^2$ if a and b are real. In this case

$$f_{2m}(x) = (x^2 + h^{2m-1})^m, \quad f_{2m+1}(x) = (x^2 - h^{2m})^m(x - (-h)^m) \quad ,$$

and it can then be shown that

$$\begin{bmatrix} 2m \\ 2j \end{bmatrix} = h^{2j(m-j)}\binom{m}{j}, \quad \begin{bmatrix} 2m \\ 2j-1 \end{bmatrix} = 0 ,$$

$$\begin{bmatrix} 2m+1 \\ 2j \end{bmatrix} = (-1)^j\, h^{j(2m-2j+1)}\binom{m}{j}, \quad \begin{bmatrix} 2m+1 \\ 2j+1 \end{bmatrix} = (-1)^{j+m}\, h^{(m-j)(2j+1)}\binom{m}{j} .$$

If a and b are complex, we can let $a = \rho e^{i\theta}$ and $b = \rho e^{-i\theta}$ with $h = \rho^2$ and $\rho > 0$. Then $a^p = b^p$ implies that $p\theta = -p\theta + 2m\pi$ and hence θ is a rational multiple $m\pi/p$ of π. Let $m/p = c/d$ with c and d relatively prime and $d > 0$. Then a/ρ and b/ρ are d-th roots of 1 if c is even and d-th roots of -1 if c is odd. The roots $a^{k-j}b^{j-1}$ of $f_k(x)$ are now of the form $\rho^{k-1}e^{(k-1-2j)\theta i}$. If $k > d$, these roots repeat in blocks of d as j varies from 1 to k. Let $k = qd + r$ with q and r integers and $0 \le r < d$. Then

$$(15) \quad f_k(x) = (-1)^{cqr}\rho^{qdr} f_r([-1\ ]^{cq}x/\rho^{qd})\left[x^d - (-1)^{c(k-1)}\rho^{(k-1)d}\right]^q \quad .$$

Now let $j = q'd + r'$ with q' and r' integers and $0 \le r' < d$. It then follows from (15) that

$$\begin{bmatrix} k \\ j \end{bmatrix} = (-1)^e h^f \binom{q}{q'} \begin{bmatrix} r \\ r' \end{bmatrix}$$

where $e = q'(d + cr + cqd + c + 1) + cqr'$ and

$$2f = d^2\left[qq' - (q')^2\right] + d(qr' + q'r - 2q'r') \quad .$$

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the last digit repeats on a period of 781, the second to last digit has a period of 3900, and the

Hexanacci Series

1, 1, 1, 1, 1, 1, 6, 11, 21, 41, 81, 161, 321, 636, 1261, 2501, 4961, 9841...

the last digit as can easily be seen above repeats on a period of 7, the sequence being:

61111116111111611111161111116...

the second to last digit however has the somewhat larger period of 7280.

Finally, for sometime, I have wanted to apply these observations on the periodicity of the last digits to some other Fibonacci problems. So far, I have only the somewhat lame observation that the Prime-Fibonacci-Number Density (that is the ratio between the number of Fibonacci numbers which are prime below a given number n and that number n) is less than

$$4/15 \int_2^x dx/\ln x \ .$$

This observation follows from the theorem that if a Fibonacci number is prime, then its subscript is prime. Thus if all Fibonacci numbers with prime subscripts were prime the density would be Euler's famous expression

$$\pi(n) = \int_2^x dx/\ln x \ .$$

However, a good number of Fibonacci Numbers are <u>not</u> prime but do have prime subscripts, some of these numbers can now be excluded from the prime-density considerations because every prime greater than 3 must end in a 1, 3, 7, or 9 and can be expressed as $6x\pm1$. Now consider the sequence of the last digit of the Fibonacci series:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1	2	3	5	8	3	1	4	5	9	4	3	7	0	7	7	4	1	5
*						*				*		*				*		*	
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
6	1	7	8	5	3	8	1	9	0	9	9	8	7	5	2	7	9	6	5
		*						*		*						*			
41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
1	6	7	3	0	3	3	6	9	5	4	9	3	2	5	7	2	9	1	0
*		*				*		*				*						*	

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