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THE NONEXISTENCE OF QUASIPERFECT NUMBERS OF CERTAIN FORMS

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1. INTRODUCTION

A natural number n is called perfect, multiperfect, or quasiperfect according as $\sigma(n) = 2n$, $\sigma(n) = kn$ ($k \geq 2$, an integer), or $\sigma(n) = 2n+1$, respectively, where $\sigma(n)$ is the sum of the positive divisors of n.

No odd multiperfect numbers are known. In many papers concerned with odd perfect numbers (summarized in McDaniel & Hagis [5]), values have been obtained which cannot be taken by the even exponents on the prime factors of such numbers, if all those exponents are equal. McDaniel [4] has given results of a similar nature for odd multiperfect numbers.

No quasiperfect numbers have been found. It is known [Cattaneo [1]) that if there are any they must be odd perfect squares, and it has recently been shown by Hagis & Cohen [3] that such a number must have at least seven distinct prime factors and must exceed 10^{35}. In this paper we shall give results analogous to those described for odd multiperfect numbers, but with extra generality. In particular, we shall show that no perfect fourth power is quasiperfect, and no perfect sixth power, prime to 3, is quasiperfect. We are unable to prove the nonexistence of quasiperfect numbers of the form m^2, where m is squarefree, but will show that any such numbers must have more than 230,000 distinct prime factors, so the chance of finding any is slight!

All italicized letters here denote nonnegative integers, with p and q primes, $p > 2$.

2. SOME LEMMAS

The following result is due to Cattaneo [1].

LEMMA 1: If n is quasiperfect and $r | \sigma(n)$, then $r \equiv 1$ or $3 \pmod 8$.

We shall need

LEMMA 2: Suppose n is quasiperfect and $p^{2a} \| n$. If $q | 2a + 1$, then

$$(q - 1)(p + 1) \equiv 0 \text{ or } 4 \pmod{16}.$$

PROOF: Notice first that if b is odd, then, modulo 8,

$$(1) \qquad \sigma(p^{b-1}) = 1 + p + p^2 + \cdots + p^{b-1} \equiv 1 + (p + 1) + \cdots + (p + 1) = 1 + \tfrac{1}{2}(b - 1)(p + 1).$$

Let $F_d(\xi)$ denote the cyclotomic polynomial of order d. It is well known that

$$\xi^m - 1 = \prod_{d | m} F_d(\xi) \qquad (m > 0),$$

so

$$(2) \qquad \sigma(p^{2\ }) = \prod_{\substack{d | 2a+1 \\ d > 1}} F_d(p).$$

Hence $\sigma(p^{q-1}) = F_q(p)\,|\,\sigma(p^{2a})\,|\,\sigma(n)$, since $\sigma(n)$ is multiplicative. From (1) and Lemma 1, $1 + \frac{1}{2}(q - 1)(p + 1) \equiv 1$ or $3 \pmod 8$ and the result follows.

LEMMA 3: If n is quasiperfect and $p^{2a}\|n$, where $a \equiv 1 \pmod 3$, then $p \not\equiv 3$ or $5 \pmod 8$ and $p \not\equiv b$ or $c \pmod q$ for b, c, q in Table 1.

Table 1

b	c	q	b	c	q	b	c	q	b	c	q
2	4	7	48	132	181	171	267	439	72	678	751
3	9	13	92	106	199	21	441	463	27	729	757
5	25	31	39	183	223	232	254	487	174	648	823
10	26	37	94	134	229	129	411	541	125	703	829
13	47	61	28	242	271	210	396	607	220	632	853
23	55	79	116	160	277	65	547	613	282	594	877
46	56	103	122	226	349	43	587	631	52	866	919
45	63	109	83	283	367	296	364	661	142	824	967
19	107	127	88	284	373	227	481	709	113	877	991
32	118	151	34	362	397	281	445	727	304	692	997
12	144	157	20	400	421	307	425	733			

PROOF: Since $a \equiv 1 \pmod 3$, we take $q = 3$ in Lemma 2 to see that $p \equiv 1$ or $7 \pmod 8$. If $p \equiv b$ or $c \pmod q$, for any triple (b, c, q) in Table 1, then

$$\sigma(p^2) \equiv 0 \pmod q.$$

From (2), we have

$$\sigma(p^{2a}) = F_3(p) \prod_{\substack{d \mid 2a+1 \\ d \geq 5}} F_d(p),$$

so

$$q\,|\,\sigma(p^2) = F_3(p)\,|\,\alpha(p^{2a})\,|\,\sigma(n).$$

But $q \equiv 5$ or $7 \pmod 8$, so Lemma 1 is contradicted. Hence $p \not\equiv b$ or $c \pmod q$.

Note: The primes q in Table 1 are all primes less than 1000 that are congruent to 5 or 7 (mod 8) and to 1 (mod 3), and b and $c \equiv b^2 \pmod q$ are the positive integers belonging to the exponent 3 (mod q). Lemma 3 provides a useful screening of primes p such that p^2 [or p^{2a} where $a \equiv 1 \pmod 3$] can exactly divide a quasiperfect number: the three smallest primes p such that $\sigma(p^2)$ has a divisor congruent to 5 or 7 (mod 8) and not eliminated by Lemma 3 are 2351, 3767, and 5431.

3. THE THEOREMS

THEOREM 1: (i) No number of the form m^4 is quasiperfect.

(ii) No number of the form m^6, where $(m, 3) = 1$, is quasiperfect.

PROOF: (i) Suppose

$$n = \prod_{i=1}^{t} p_i^{4a_i}$$

and that n is quasiperfect. It is easy to see that $\sigma(p_i^{4a_i}) \equiv 1 \pmod 4$ so, by Lemma 1, $\sigma(p_i^{4a_i}) \equiv 1 \pmod 8$ and hence

$$\sigma(n) = \prod_{i=1}^{t} \sigma(p_i^{4a_i}) \equiv 1 \pmod 8.$$

But n is an odd square, so $\sigma(n) = 2n + 1 \equiv 3 \pmod 8$, a contradiction.

(ii) Suppose
$$n = \prod_{i=1}^{t} p_i^{6a_i}, \; p_i \geq 5,$$
and that n is quasiperfect. We have $\sigma(p_i^{6a_i}) \equiv 1 \pmod 3$, so
$$\sigma(n) = \prod_{i=1}^{t} \sigma(p_i^{6a_i}) \equiv 1 \pmod 3.$$
However, since n is a square and $3 \nmid n$, we have $\sigma(n) = 2n + 1 \equiv 0 \pmod 3$, another contradiction.

THEOREM 2: If a number of the form $\prod_{i=1}^{t} p_i^{24a_i+2b}$ is quasiperfect, then $b = 1$, 5, or 11, $p_i \not\equiv 3 \pmod 8$ for any i, and $t \geq 10$.

PROOF: Suppose
$$n = \prod_{i=1}^{t} p_i^{24a_i+2b}$$
is quasiperfect. From Theorem 1(i), $b \neq 0$, 2, 4, 6, 8, or 10. It then follows, using (1), that $p_i \not\equiv 3 \pmod 8$ for any i (or see Cattaneo [1]). In particular, $3 \nmid n$, so from Theorem 1(ii), $b \neq 3$ or 9. Suppose $b = 7$. Then by Lemma 3 we have $p_i \equiv 1$ or $7 \pmod 8$ for all i. If $p_i \equiv 1 \pmod 8$ for some i, then
$$\sigma(p_i^{24a_i+14}) \equiv 24a_i + 15 \equiv 7 \pmod 8,$$
and this contradicts Lemma 1. Hence, for all i, $p_i \equiv 7 \pmod 8$, so
$$\sigma(p_i^{24a_i+14}) \equiv 1 \pmod 8 \quad \text{and} \quad \sigma(n) \equiv 1 \pmod 8.$$
As in the proof of Theorem 1(i), this is a contradiction. Thus $b \neq 7$. Since $p_i \not\equiv 3 \pmod 8$ for any i, we have, finally, if $t \leq 9$,
$$\frac{\sigma(n)}{n} < \prod_{i=1}^{t} \frac{p_i}{p_i - 1} \leq \frac{5}{4}\,\frac{7}{6}\,\frac{13}{12}\,\frac{17}{16}\,\frac{23}{22}\,\frac{29}{28}\,\frac{31}{30}\,\frac{37}{36}\,\frac{41}{40} < 2.$$
This contradicts the fact that $\sigma(n)/n = 2 + 1/n > 2$. Hence $t \geq 10$.

We are unable to establish in particular that there are no quasiperfect numbers of the form
$$\prod_{i=1}^{t} p_i^{2b} \text{ for } b = 1, 5, \text{ or } 11.$$
The next theorem includes information on the case $b = 1$, and the final theorem is related to the cases $b = 5$ and $b = 11$.

THEOREM 3: If a number of the form $\prod_{i=1}^{t} p_i^{6a_i+2}$ is quasiperfect, then $t \geq 230876$.

PROOF: Let q_i be the ith prime ($q_1 = 2$), and let Π' denote a product over primes not congruent to 3 or 5 (mod 8) or to b or c (mod q), with b, c, q as in Lemma 3. The 5000th such prime is $P = 309769 = q_{26775}$ and the 225876th prime greater than P is $Q = 3538411$. We have computed that
$$\prod_{q_i \leq P}' \frac{q_i}{q_i - 1} < 1.6768 = \alpha, \text{ say.}$$

Suppose $n = \prod_{i=1}^{t} p^{6a_i+2}$ is quasiperfect and that $t < s = 230876$. From Lemma 3, $p_i \not\equiv$ 3 or 5 (mod 8) and $p_i \not\equiv b$ or c (mod q) for any i and any triple (b, c, q) in Lemma 3. Thus

$$\frac{\sigma(n)}{n} \le {\prod_{q_i \le P}}' \frac{\sigma(q_i^{6a_i+2})}{q_i^{6a_i+2}} \prod_{i=5001}^{t} \frac{\sigma(p_i^{6a_i+2})}{p_i^{6a_i+2}} < {\prod_{q_i \le P}}' \frac{q_i}{q_i - 1} \prod_{i=26776}^{s+21775} \frac{q_i}{q_i - 1} < \alpha \prod_{P < q \le Q} \frac{q}{q - 1}$$

$$= \alpha \prod_{q \le Q} \frac{q}{q - 1} \prod_{q \le P} \frac{q - 1}{q} < \frac{\alpha}{\log P} \log\left(Q + \frac{2}{\sqrt{Q}}\right) < 2,$$

using Theorem 23 in Rosser & Schoenfeld [6]. But n is quasiperfect, so $\sigma(n)/n > 2$, and we have a contradiction. Hence $t \ge s$.

THEOREM 4: No number of the form $3^{2a}m^{2b}$, where $3 \nmid m$, $a \equiv 2 \pmod 5$, and either $b \equiv 0 \pmod 5$ or $b \equiv 0 \pmod{11}$ is quasiperfect.

PROOF: Suppose $n = 3^{2a}m^{2b}\ (3 \nmid m)$, with a, b as given, is quasiperfect. Since

$$\sigma(3^{2a}) = \prod_{\substack{d|2a+1 \\ d>1}} F_d(3) = F_5(3) \prod_{\substack{d|2a+1 \\ d>1,\ d\ne 5}} F_d(3),$$

we have $11^2 = F_5(3)\,|\,\sigma(3^{2a})\,|\,\sigma(n)$. Since $3^{10} \equiv 1 \pmod{121}$,

$$\sigma(n) = 2n + 1 \equiv 2 \cdot 3^4 m^{2b} + 1 \equiv 0 \pmod{121}.$$

From this, $m^{2b} \equiv 59 \pmod{121}$, and thus

$$m^b \equiv 46 \text{ or } 75 \pmod{121}. \tag{3}$$

For each possible value of $c = \phi(121)/(b, \phi(121))$ it is not the case that $46^c \equiv 1 \pmod{121}$ or $75^c \equiv 1 \pmod{121}$ (ϕ is Euler's function). Euler's criterion for the existence of power residues (Griffin [2, p. 129]) shows the congruences (3) to be insolvable. This contradiction proves the theorem.

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