ELEMENTARY PROBLEMS AND SOLUTIONS

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Please send all communications regarding ELEMENTARY PROBLEMS AND SOLUTIONS to PROFESSOR A. P. HILLMAN, 709 Solano Dr., S.E.; Albuquerque, NM 87108. Each solution or problem should be on a separate sheet (or sheets). Preference will be given to those typed with double spacing in the format used below. Solutions should be received within four months of the publication date.

DEFINITIONS

The Fibonacci numbers \( F_n \) and the Lucas numbers \( L_n \) satisfy
\[
F_{n+2} = F_{n+1} + F_n, \quad F_0 = 0, \quad F_1 = 1
\]
and
\[
L_{n+2} = L_{n+1} + L_n, \quad L_0 = 2, \quad L_1 = 1.
\]
Also, \( a \) and \( b \) designate the roots \( (1 + \sqrt{5})/2 \) and \( (1 - \sqrt{5})/2 \), respectively, of
\[
x^2 - x - 1 = 0.
\]

PROBLEMS PROPOSED IN THIS ISSUE

B-466 Proposed by Herta T. Freitag, Roanoke, VA

Let \( A_n = 1 \cdot 2 - 2 \cdot 3 + 3 \cdot 4 - \ldots + (-1)^{n-1} n(n + 1) \).

(a) Determine the values of \( n \) for which \( 2A_n \) is a perfect square.
(b) Determine the values of \( n \) for which \( |A_n|/2 \) is the product of two consecutive positive integers.

B-467 Proposed by Herta T. Freitag, Roanoke, VA

Let \( A_n \) be as in B-466 and let \( B_n = \sum_{k=1}^{n} \sum_{i=1}^{k} k \). For which positive integers \( n \) is \( |A_n| \) an integral divisor of \( B_n \)?

B-468 Proposed by Miha'ly Bencze, Brasov, Romania

Find a closed form for the \( n \)th term \( a_n \) of the sequence for which \( a_1 \) and \( a_2 \) are arbitrary real numbers in the open interval \((0, 1)\) and
\[
a_{n+2} = a_{n+1} \sqrt{1 - a_n^2} + a_n \sqrt{1 - a_{n+1}^2}.
\]
The formula for \( a_n \) should involve Fibonacci numbers if possible.

B-469 Proposed by Charles R. Wall, Trident Technical College, Charleston, SC

Describe the appearance in base \( F_n \) notation of:
(a) \( 1/F_{n-1} \) for \( n \geq 5 \); (b) \( 1/F_{n+1} \) for \( n \geq 3 \).

B-470 Proposed by Larry Taylor, Rego Park, NY

Find positive integers \( a, b, c, r, \) and \( s \) and choose each of \( C_n, \overline{H}_n, I_n \) to be \( F_n \) or \( L_n \) so that \( aC_n, b\overline{H}_{n+r}, cI_{n+s} \) are in arithmetic progression for \( n \geq 0 \) and this progression is 6, 6, 6 for some \( n \).
**B-471** Proposed by Larry Taylor, Rego Park, NY

Do there exist positive integers \( d \) and \( t \) such that \( aG_n, bH_{n+r}, \alpha I_{n+s}, dJ_{n+t} \) are in arithmetic progression, with \( e_n \) equal to \( F_n \) or \( L_n \) and everything else as in \( B-470 \)?

**Solutions**

Lucas Analogue of Cosine Identity

**B-442** Proposed by P. L. Mana, Albuquerque, NM

The identity \( 2 \cos^2 \theta = 1 + \cos(2\theta) \) leads to the identity

\[
8 \cos^8 \theta = 3 + 4 \cos(2\theta) + \cos(4\theta).
\]

Are there corresponding identities on Lucas numbers?

Solution by Sahib Singh, Clarion State College, Clarion, PA

Yes; \( L_{2n} = a^n + b^n = (a^n + b^n)^2 - 2(-1)^n = L_n^2 - 2(-1)^n \). Hence

\[(1)\quad L_n^2 = L_{2n} + 2(-1)^n.\]

Using (1), \( L_{2n}^2 = L_{4n} + 2 \). Again using (1), the above equation reduces to

\[
(b^n - 2(-1)^n)^2 = L_{4n} + 2,
\]

which yields

\[(2)\quad L_n^4 = 6 + 4(-1)^n L_{2n} + L_{4n}.\]

Equations (1) and (2) are the required identities.


Lucas Products Identity

**B-443** Proposed by Gregory Wulczyn, Bucknell University, Lewisburg, PA

For all integers \( n \) and \( \omega \) with \( \omega \) odd, establish the following:

\[
L_{n+2\omega}L_{n+\omega} - 2L_{2\omega}L_{n+\omega}L_{n-\omega} - L_{n-\omega}L_{n-2\omega} = L_{3\omega}^2(L_{3\omega} - 2L_{\omega}).
\]

Solution by Sahib Singh, Clarion State College, Clarion, PA

The given equation is equivalent to:

\[
L_{n+2\omega}L_{n+\omega} - L_{n-\omega}L_{n-2\omega} = L_{3\omega}^2 = 2L_{\omega}(L_{n+\omega}L_{n-\omega} - L_{n}^2).
\]

Using the identity \( L_{n+\omega}L_{n-\omega} - L_n^2 = 5(-1)^{n+\omega} \), the above equation becomes:

\[(1)\quad L_{n+2\omega}L_{n+\omega} - L_{n-\omega}L_{n-2\omega} = 10(-1)^{n+\omega} L_{\omega}^2 L_{3\omega}^2.
\]

Using \( L_n = a^n + b^n \), the left side of (1) becomes

\[
2(-1)^{n+\omega}(L_{\omega} + L_{3\omega}).
\]

Thus (1) reduces to \( L_{\omega} + L_{3\omega} = 5L_{\omega}^2 L_{3\omega}^2 \). Since \( \omega \) is odd, by using \( L_n = a^n + b^n \), the above equation is true and we are done.

Also solved by Paul S. Bruckman, Herta T Freitag, Calvin L. Gardner, Bob Priellipp, M. Wachtel, and the proposer.
Generating Palindromes

B-444 Proposed by Herta T. Freitag, Roanoke, VA

In base 10, the palindromes (i.e., numbers reading the same forward or backward) 12321 and 112232211 are converted into new palindromes using

\[ 99[10^3 + 9(12321)] = 11077011 \quad \text{and} \quad 99[10^5 + 9(112232211)] = 100008800001. \]

Generalize on these to obtain a method or methods for converting certain palindromes in a general base \( b \) to other palindromes in base \( b \).

Solution by Paul S. Bruckman, Concord CA

Let \( \sigma_b \) denote the set of palindromes in base \( b \). We will prove the following theorem.

**THEOREM:** If \( m \geq 1 \), let \( P \in \sigma_b \) be given by

\[ P \equiv \sum_{k=0}^{m-1} (b^k + b^{2m-k})\theta_k \equiv (\theta_0 \theta_1 \theta_2 \ldots \theta_{m-1} \theta_m \ldots \theta_1 \theta_0). \]

Moreover, suppose the digits \( \theta_k \) satisfy the following conditions:

1. \( 1 \leq \theta_0 \leq \theta_1 \leq b-1 \), if \( m = 1 \);
2. \( 1 \leq \theta_0 \leq \theta_1 \leq \theta_2 \leq b-1 \), if \( m \geq 2 \);
3. \( 0 \leq \theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3} \leq b-1 \), if \( 3 \leq k \leq m-1 \);
4. \( 0 \leq \theta_{m-2} \leq \theta_m \leq b-1 \), if \( m \geq 3 \).

Let \( Q \equiv (b^2 - 1)(b^{n+1} + (b - 1)P) \).

Then \( Q \in \sigma_b \).

**PROOF:**

\[ Q = (b^2 - 1)(b^{n+2} + b^{n+1}) + (b^3 - b^2 - b + 1)P \]

\[ = (b - 1)(b^{n+2} + b^{n+1}) + \sum_{k=0}^{m-1} (b^{k+3} + b^{2m+3-k})\theta_k - \sum_{k=0}^{m-1} (b^{k+2} + b^{2m+2-k})\theta_k \]

\[ - \sum_{k=0}^{m-1} (b^{k+1} + b^{2m+1-k})\theta_k + \sum_{k=0}^{m+1} (b^{k+1} + b^{2m-k})\theta_k + (b^{n+3} - b^{n+2} - b^{n+1} + b^n)\theta_m \]

\[ = (b - 1)(b^{n+2} + b^{n+1}) + \sum_{k=2}^{m+2} (b^k + b^{2m+3-k})\theta_{k-3} - \sum_{k=2}^{m+1} (b^k + b^{2m+3-k})\theta_{k-2} \]

\[ - \sum_{k=1}^{m} (b^k + b^{2m+3-k})\theta_{k-1} + \sum_{k=0}^{m-1} (b^k + b^{2m+3-k})\theta_{k+1} + (b^{n+3} - b^{n+2} - b^{n+1} + b^n)\theta_m. \]

After some manipulation, this last expression simplifies to the following:

\[ Q = (b^0 + b^{2m+3})\theta_0 + (b^1 + b^{2m+2})(\theta_1 - \theta_0) + (b^2 + b^{2m+1})(\theta_2 - \theta_1 - \theta_0) \]

\[ + \sum_{k=3}^{m} (b^k + b^{2m+3-k})(\theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3}) + (b^{n+1} + b^{n+2})(b-1 - \theta_m + \theta_{n-2}). \]

If we make the following definitions

\[ \sigma_0 = \theta_0, \]
\[ \sigma_1 = \theta_1 - \theta_0, \]
\[ \sigma_2 = \theta_2 - \theta_1 - \theta_0, \]
\[ \sigma_k = \theta_k - \theta_{k-1} - \theta_{k-2} + \theta_{k-3}, \quad 3 \leq k \leq m, \]

then we have

\[ Q = \sigma_0 + \sum_{k=1}^{m} \sigma_k. \]
and

\( c_{n+1} = b - 1 - \theta_m + \theta_{m-2} \),

we may express \( Q \) as follows:

\[
Q = \sum_{k=0}^{m+1} (b^k + b^{2m+3-k})c_k.
\]

Moreover, we see from conditions (2), (3), and (4) that, for \( 0 \leq k \leq m + 1 \), the inequalities \( 0 \leq c_k \leq b-1 \) (with \( c_0 \geq 1 \)) are satisfied, i.e., the \( c_k \)'s are digits in base \( b \). Therefore, \( Q = (c_0c_1c_2 \ldots c_m)_{b} \) \( \in \Theta_b \). Q.E.D.

This result is readily specialized to decimal numbers by setting \( b = 10 \) in the theorem. Thus, if \( P \in \Theta_{10} \) is given by (1)-(4), with \( b = 10 \), then

\[
Q = 99(10^{m+1} + 9P)
\]

is a palindrome in base 10.

Also solved by the proposer.

**Simple Form**

**B-445** Proposed by Wray G. Brady, Slippery Rock State College, Slippery Rock, PA

Show that

\[
5F_{2n+2}^2 + 2L_{2n}^2 + 5F_{2n-2}^2 = L_{2n+2}^2 + 10F_{2n}^2 + L_{2n-2}^2,
\]

and find a simpler form for these equal expressions.

Solution by F. D. Parker, St. Lawrence University, Canton, NY

Using the identities \( L_n = a^n + b^n \), \( F_n = (a^n - b^n)/\sqrt{5} \), and \( ab = -1 \), both sides reduce to

\[
a^{4n+4} + b^{4n+4} + 2a^{4n} + 2b^{4n} + a^{4n-4} + b^{4n-4},
\]

which can be written

\[
L_{4n+4} + 2L_{4n} + L_{4n-4}.
\]

Using \( L_n = L_{n-1} + L_{n-2} \), we see that this is equal to \( 9L_{4n} \).


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