SUMS OF CONSECUTIVE INTEGERS

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The purpose of this note is to simplify and extend the results in [1]. Given a positive integer n, let $C_e(n)$, $C_o(n)$ denote the number of representations of as a sum of an even, odd number of consecutive positive integers.

THEOREM 1: $C_o(n)$ is the number of odd divisors d of n such that $\dfrac{d(d+1)}{2} \leq n$ and $C_e(n)$ is the number of odd divisors d of n such that $\dfrac{d(d+1)}{2} > n$.

PROOF: If n is a sum of an odd number b of consecutive integers, then there exists an integer $a \geq 1$ such that

$$n = \sum_{i=0}^{b-1} (a + i) = b\left(a + \frac{b-1}{2}\right).$$

Hence b is an odd divisor of n with $\dfrac{b(b+1)}{2} \leq n$, since

$$\frac{b+1}{2} \leq a + \frac{b-1}{2} = \frac{n}{b}.$$

If b is an odd divisor of n such that $\dfrac{b(b+1)}{2} \leq n$, let $a = \dfrac{n}{b} - \dfrac{b-1}{2}$. Then $a \geq 1$ and

$$n = b\ a + \frac{b-1}{2} = \sum_{i=0}^{b-1} (a + i),$$

so that n is the sum of an odd number of consecutive positive integers.

If n is a sum of an even number b of consecutive positive integers, then there exists an integer $a \geq 1$ such that

$$n = \sum_{i=0}^{b-1} (a + i) = \frac{b}{2}(2a + b - 1).$$

Let $d = 2a + b - 1$, then d is odd, d divides n, and $\dfrac{d(d+1)}{2} > n$, since

$$d + 1 = 2a + b > b = \frac{2n}{d}.$$

If d is an odd divisor of n such that $\dfrac{d(d+1)}{2} > n$, let $b = \dfrac{2n}{d}$ and $a = \dfrac{(d+1-b)}{2}$. Then $a \geq 1$, b is even, and

$$n = \frac{bd}{2} = \frac{b}{2}(2a + b - 1) = \sum_{i=0}^{b-1} (a + i),$$

so that n is a sum of an even number of consecutive positive integers. □

An immediate consequence of Theorem 1 is the following corollary.

COROLLARY 1: Let $n = 2\ m$, $r \geq 0$, m odd. The number of representations of n as a sum of consecutive positive numbers is $\tau(m)$ (the number of divisors of m). □

This result is also in [2], which of course gives the results in [1].

We also find a characterization of primes.

COROLLARY 2: Let n be an odd positive integer. Then n is composite if and only if there is a pair of positive numbers u, v such that

$$8n = u^2 - v^2;\ u - v \geq 6. \tag{1}$$

PROOF: If n is odd composite, then n is the sum of at least three consecutive integers by Theorem 1. That is

$$n = a + (a + 1) + \cdots + (a + k),\ k \geq 2.$$

Hence $2n = (k + 1)(2a + k)$. Let $v = 2a - 1$ and $u = 2k + 2a + 1$. Then

$$k + 1 = \frac{u - v}{2} \quad \text{and} \quad 2a + k = \frac{u + v}{2},$$

so that $8n = u^2 - v^2$ and $u - v \geq 6$. Note that u, v are odd. Conversely, given an odd integer n satisfying (1), we find

$$8n = (u + v)(u - v).$$

If n is prime and $u - v$ is even, then $u - v = 8$, $2n$, or $4n$. When $u - v = 8$, we have $2u = n + 8$ so that $n = 2$, while $u - v = 2n$ implies that $u = 2 + n$, and hence $v = 2 - n \leq 0$. If $u - v = 4n$, then $u + v = 2$ and $u = v = 1$, which says that $n = 0$. Thus, if n is a prime, we must have $u + v = 8$ and $u - v = n$, which implies once again that $n = 2$.

We conclude that n must be composite. It is also simple to solve the above system for a and k. □

It is not easy to find $C_o(n)$ explicitly. For instance, let $\tau_o(n, x)$ denote the number of odd positive divisors of n which are $\leq x$. One finds

$$\tau_o(n, x) = \sum_{\substack{d \leq x \\ d \text{ odd}}} \frac{c_d(n)}{d} \sum_{\substack{k \leq x/d \\ k \text{ odd}}} \frac{1}{k},$$

where $c_d(n)$ is the Ramanujan function. This is not altogether satisfactory, but it will yield an estimate. One direct but very elaborate way to find $\tau_o(n, x)$ explicitly is by counting lattice points as follows. Write $n = 2^{a_0} p_1^{a_1} \ldots p_k^{a_k}$ as a product of primes. An odd divisor d of n is of the form $d = p_1^{b_1} \ldots p_k^{b_k}$, where $0 \leq b_i \leq a_i$. The inequality $d \leq x$ means

$$b_1 \log p_1 + \cdots + b_k \log p_k \leq \log x.$$

Let $e_1, \ldots, e_k$ be the standard basis of $\mathbb{R}^k$. Consider the parallel-piped P determined by $a_1 e_1, \ldots, a_k e_k$ and the hyperplane H with equation

$$x_1 \log p_1 + \cdots + x_k \log p_k = \log x.$$

Then $\tau_o(n, x)$ is the number of lattice points in the region "below" H which are also contained in P. There are of course k^2 possible intersections of H with P to consider, a formidable task! However, we have, perhaps a little surprisingly,

COROLLARY 3: Write $n = 2^k m$, where m is odd. Then

$$C_o(n) \quad \frac{1}{2}\tau(m);\ C_1(n) \leq \left(k + \frac{1}{2}\right)\tau(m).$$

In particular, when n is odd, we have

$$C_e(n) \leq \frac{\tau(n)}{2} \leq C_o(n).$$

PROOF: It is very easy to show that

$$\sqrt{n} \leq \frac{-1 + \sqrt{1 + 8n}}{2}, \tag{1}$$

and if $d > 0$, then

$$\frac{d(d+1)}{2} \le n \Longleftrightarrow d \le \frac{-1+\sqrt{1+8n}}{2} \tag{2}$$

Thus $C_0(n)$ is at least the number of odd divisors d of n that are $\le \sqrt{n}$, so *a fortiori* we have

$$C_0(n) \ge \tau_0(m, \sqrt{m}).$$

If $d|m$ and $d \le \sqrt{m}$, then $m/d|m$ and $m|d \ge \sqrt{m}$. Thus

$$\tau(m, \sqrt{m}) = \begin{cases} \dfrac{\tau(m)}{2} & \text{if } m \text{ is not a square} \\ \dfrac{\tau(m)+1}{2} & \text{if } m \text{ is a square.} \end{cases}$$

Hence $C_0(n) \ge \tau(m)/2$. We have $C_1(n) = \tau(n) - C_0(n)$, and thus

$$C_1(n) \le (k+1)\tau(m) - \frac{\tau(m)}{2} = \left(k + \frac{1}{2}\right)\tau(m).$$

This completes the proof. □

REFERENCES

1. B. de La Rosa. "Primes, Powers, and Partitions." *The Fibonacci Quarterly* 16, no 6 (1978):518-22.
2. W. J. Leveque. "On Representation as a Sum of Consecutive Integers." *Canad. J. Math.* 4 (1950):399-405.

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CONCERNING A PAPER BY L. G. WILSON

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1. INTRODUCTION

Wilson [3] uses the expression (2.1) below, which approximates the Fibonacci and Lucas sequences $\{F_r\}$ and $\{L_r\}$, respectively, for r sufficiently large. The object of this paper is to make known this and another expression (3.1) by applying techniques different from those used in [3]. In particular, we need

$$\beta_i = 4\cos^2\frac{i\pi}{2n}. \tag{1.1}$$

Special attention is directed to the sequence (2.4).

2. A GENERATING EXPRESSION

Consider

$$F_r(x, y) \equiv T_r = \left(\frac{x+\sqrt{x^2+4x}}{2}\right)^{r-1} y^{-1/2}, \tag{2.1}$$