REMARK 2: There are 51 solutions of $P_x^3 = P_s^3 P_y^3$ with $P_x < 10^6$. There are 43 solutions of $P_x^n = P_s^n P_y^n$ with $5 \leq n \leq 36$ and $P_x^n < 10^6$. In just two of these, $x = P_s$:

$$P_{477}^5 = P_{18}^5 P_{22}^5 \quad \text{and} \quad P_{946}^6 = P_{22}^6 P_{31}^6.$$

For $36 \leq n \leq 720$, there are no solutions with $P_x^n < 10^6$.

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WAITING FOR THE *K*TH CONSECUTIVE SUCCESS AND THE FIBONACCI SEQUENCE OF ORDER *K*

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1. INTRODUCTION AND SUMMARY

In the sequel, k is a fixed integer greater than or equal to 2, and n is an integer as specified. Let N_k be a random variable denoting the number of trials until the occurrence of the kth consecutive success in independent trials with constant success probability p $(0 < p < 1)$. Shane [6] and Turner [7] considered the problem of obtaining the distribution of N_k. The first author found a formula for $P[N_k = n]$ $(n \geq k)$, as well as for $P[N_k \leq x]$ $(x \geq k)$, in terms of the polynacci polynomials of order k in p. Turner derived a formula for $P[N_k = n + k - 1]$ $(n \geq 1)$ in terms of the entries of the Pascal-T triangle. Both Shane and Turner first treated the special cases $p = 1/2$, $k = 2$, and $p = 1/2$, general k. For these cases, their formulas coincide.

Presently, we reconsider the problem and derive a new and simpler formula for $P[N_k = n + k]$ $(n \geq 0)$, in terms of the multinomial coefficients (see Theorem 3.1). The method of proof is also new. Interestingly enough, our formula includes as corollaries the special formulas of Shane and Turner. We present these results in Section 3. In Section 2, we obtain an expansion of the Fibonacci sequence of order k in terms of the multinomial coefficients (see Theorem 2.1), which is of interest in its own right and instrumental in deriving one of the corollaries.

2. THE FIBONACCI SEQUENCE OF ORDER *K*

In this section, we consider the Fibonacci sequence of order k and derive an expansion of it, in terms of the multinomial coefficients.

DEFINITION 1: The sequence $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ is said to be the Fibonacci sequence of order k if $f_0^{(k)} = 0$, $f_1^{(k)} = 1$, and

$$f_n^{(k)} = \begin{cases} f_{n-1}^{(k)} + \cdots + f_1^{(k)} & \text{if } 2 \le n \le k \\ f_{n-1}^{(k)} + \cdots + f_{n-k}^{(k)} & \text{if } n \ge k+1. \end{cases} \tag{2.1}$$

Turner [7] calls $\left\{f_n^{(k)}\right\}_{n=1}^{\infty}$ the kth-order Fibonacci-T sequence. With $f_n^{(k)} = 0$ $(n \le -1)$, $\left\{f_n^{(k)}\right\}_{n=-\infty}^{\infty}$ is called by Gabai [4] the Fibonacci k-sequence. The shift version of the last sequence, obtained by setting $F_{n,k} = f_{n+1}^{(k)}$, is called by Shane [6] the polynacci sequence of order k. See, also, Fisher and Kohlbecker [3], and Hoggatt [5].

Denoting by F_n and T_n, as usual, the Fibonacci and Tribonacci numbers, respectively, it follows from (2.1) that

$$f_n^{(2)} = F_n \quad \text{and} \quad f_n^{(3)} = T_n,\ n \ge 0. \tag{2.2}$$

The Tribonacci numbers seem to have been introduced by Agronomoff [1]. Their name, however, is due to Feinberg [2], who rediscovered them.

We now proceed to establish the following lemma.

LEMMA 2.1: Let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, and assume that

$$c_n^{(k)} = \begin{cases} 1 & \text{if } n = 0,\ 1 \\ 2c_{n-1}^{(k)} & \text{if } 2 \le n \le k \\ 2c_{n-1}^{(k)} - c_{n-1-k}^{(k)} & \text{if } n \ge k+1. \end{cases} \tag{2.3}$$

Then

$$c_n^{(k)} = f_{n+1}^{(k)},\ n \ge 0.$$

PROOF: From (2.1) and (2.3), it follows that

$$c_n^{(k)} = f_{n+1}^{(k)},\ 0 \le n \le k. \tag{2.4}$$

Suppose next that

$$c_n^{(k)} = f_{n+1}^{(k)},\ k+1 \le n \le m, \tag{2.5}$$

for some integer $m \ge k+1$. Then

$$\begin{aligned} c_{m+1}^{(k)} &= 2c_m^{(k)} - c_{m-k}^{(k)}, \text{ by (2.3)}, \\ &= 2f_{m+1}^{(k)} - f_{m+1-k}^{(k)}, \text{ by (2.4) and (2.5)}, \\ &= f_{m+2}^{(k)}, \text{ by (2.1)}. \end{aligned} \tag{2.6}$$

Relations (2.4)-(2.6) show the lemma.

We will employ Lemma 2.1 to prove the following lemma.

LEMMA 2.2: Let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k, and denote by $A_n^{(k)}$ the number of arrangements of $n + k$ elements (f or s), such that no k adjacent ones are all s, but the last k. Then

$$A_n^{(k)} = f_{n+1}^{(k)},\ n \ge 0.$$

PROOF: For each $n \ge 0$, define $a_n^{(k)}(f)$, $a_n^{(k)}(s)$, and $a_n^{(k)}$ as follows:

(2.7) $a_n^{(k)}(f)$ = the number of arrangements of n elements (f or s), such that the last is f and no k adjacent ones are all s $(n \ge 1)$; $a_0^{(k)}(f) = 1$.

(2.8) $a_n^{(k)}(s)$ = the number of arrangements of n elements (f or s), such that the last is s and no k adjacent ones are all s $(n \ge 1)$; $a_0^{(k)}(s) = 0$.

(2.9) $a_n^{(k)}$ = the number of arrangements of n elements (f or s), such that no k adjacent ones are all s $(n \geq 1)$; $a_0^{(k)} = 1$.

Relations (2.7)-(2.9) imply

$$a_n^{(k)} = a_n^{(k)}(f) + a_n^{(k)}(s), \quad n \geq 0, \tag{2.10}$$

$$a_{n+1}^{(k)}(f) = a_n^{(k)}, \quad n \geq 0, \tag{2.11}$$

and

$$a_{n+1}^{(k)}(s) = \begin{cases} a_n^{(k)}, & 0 \leq n \leq k-2 \\ a_n^{(k)} - a_{n+1-k}^{(k)}(f), & n \geq k-1. \end{cases} \tag{2.12}$$

The second part of (2.12) is due to the fact that, for $n \geq k - 1$, $a_{n+1}^{(k)}(s)$ equals $a_n^{(k)}$ minus the number of arrangements among the $a_n^{(k)}(s)$ whose last $k - 1$ elements are all s. Adding (2.11) and (2.12) and utilizing (2.10) and (2.11), we obtain

$$a_{n+2}^{(k)}(f) = \begin{cases} 2a_{n+1}^{(k)}(f), & 0 \leq n \leq k-2 \\ 2a_{n+1}^{(k)}(f) - a_{n+1-k}^{(k)}(f), & n \geq k-1. \end{cases} \tag{2.13}$$

We also have, by the definition of $A_n^{(k)}$ and (2.7), that

$$a_n^{(k)}(f) = A_n^{(k)}, \quad n \geq 0, \tag{2.14}$$

with $A_0^{(k)} = A_1^{(k)} = 1$. The last two relations give

$$A_n^{(k)} = \begin{cases} 1, & n = 0,\ 1 \\ 2A_{n-1}^{(k)}, & 2 \leq n \leq k \\ 2A_{n-1}^{(k)} - A_{n-1-k}^{(k)}, & n \geq k+1, \end{cases}$$

which establishes Lemma 2.2, by means of Lemma 2.1

The following theorem gives a formula for the Fibonacci numbers of order k in terms of the multinomial coefficients.

THEOREM 2.1: Let $\left\{f_n^{(k)}\right\}_{n=0}^{\infty}$ be the Fibonacci sequence of order k. Then

$$f_{n+1}^{(k)} = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \quad n \geq 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$ satisfying the relation $n_1 + 2n_2 + \cdots + kn_k = n$.

PROOF: By Lemma 2.2 and (2.14),

$$f_{n+1}^{(k)} = a_n^{(k)}(f), \quad n \geq 0. \tag{2.15}$$

Next, observe that an arrangement of n elements (f or s) is one of the $a_n^{(k)}(f)$ if and only if n_1 of its elements are $e_1 = f$; n_2 of its elements are $e_2 = sf, \ldots$; n_k of its elements are $e_k = \underbrace{ss \ldots s}_{k-1}f$ $(n_1 + 2n_2 + \cdots + kn_k = n)$. Now, for fixed nonnegative integers $n_1, \ldots, n_k$, the number of arrangements of the $n_1 + \cdots + n_k$ e's is $\binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}$, However, $n_1, \ldots, n_k$ are allowed to vary, subject to the condition $n_1 + 2n_2 + \cdots + kn_k = n$. Therefore,

$$a_n^{(k)}(f) = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}, \quad n \geq 0, \tag{2.16}$$

where the summation is taken over all nonnegative integers $n_1, \ldots, n_k$, such that $n_1 + 2n_2 + \cdots + kn_k = n$. Relations (2.15) and (2.16) establish the theorem.

Setting $k = 2$ and $k = 3$ in Theorem 2.1, and defining $[x]$ to be the greatest integer in x, as usual, we obtain the following corollaries, respectively.

COROLLARY 2.1: Let $\left\{F_n\right\}_{n=0}^{\infty}$ be the Fibonacci sequence. Then

$$F_{n+1} = \sum_{i=0}^{[n/2]} \binom{n-i}{i}, \quad n \geq 0.$$

COROLLARY 2.2: Let $\left\{T_n\right\}_{n=0}^{\infty}$ be the Tribonacci sequence. Then

$$T_{n+1} = \sum_{i=0}^{[n/2]} \sum_{j=0}^{[(n-2i)/3]} \binom{i+j}{i}\binom{n-i-2j}{i+j}, \quad n \geq 0.$$

The first result is well known. The second, however, does not appear to have been noticed.

3. WAITING FOR THE KTH CONSECUTIVE SUCCESS

In this section we state and prove Theorem 3.1, which expresses $P[N_k = n + k]$ $(n \geq 0)$ in terms of the multinomial coefficients. We also give two corollaries of the theorem, which re-establish all the special formulas of Shane [6] and Turner [7] for the probability density function of N_k.

THEOREM 3.1: Let N_k be a random variable denoting the number of trials until the occurrence of the kth consecutive success in independent trials with success probability p $(0 < p < 1)$. Then

$$P[N_k = n + k] = \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} (1-p)^{n_1 + \cdots + n_k} p^{n+k-(n_1 + \cdots + n_k)}, \quad n \geq 0,$$

where the summation is over all nonnegative integers $n_1, \ldots, n_k$, such that $n_1 + 2n_2 + \cdots + kn_k = n$.

PROOF: A typical element of the event $[N_k = n + k]$ is an arrangement

$$x_1 x_2 \ldots x_{n_1 + \cdots + n_k} \underbrace{ss \ldots s}_{k},$$

such that n_1 of the x's are $e_1 = f$; n_2 of the x's are $e_2 = sf$, ...; n_k of the x's are $e_k = \underbrace{ss \ldots s}_{k-1}f$, and $n_1 + 2n_2 + \cdots + kn_k = n$. Fix $n_1, \ldots, n_k$. Then the number of the above arrangements is $\binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k}$, and each one of them has probability

$$P\left[x_1 x_2 \ldots x_{n_1 + \cdots + n_k} \underbrace{ss \ldots s}_{k}\right] = \left[P\{e_1\}\right]^{n_1} \left[P\{e_2\}\right]^{n_2} \ldots \left[P\{e_k\}\right]^{n_k} P\{\underbrace{ss \ldots s}_{k}\}$$
$$= (1-p)^{n_1 + \cdots + n_k} p^{n+k-(n_1 + \cdots + n_k)}, \quad n \geq 0,$$

by the independence of the trials, the definition of e_j $(1 \leq j \leq k)$, and $P\{s\} = p$. Therefore,

$$P\left[\text{all } x_1 x_2 \ldots x_{n_1 + \cdots + n_k} \underbrace{ss \ldots s}_{k};\ n_j \geq 0 \text{ fixed } (1 \leq j \leq k),\ \sum_{j=1}^{k} jn_j = n\right]$$
$$= \binom{n_1 + \cdots + n_k}{n_1, \ldots n_k} (1-p)^{n_1 + \cdots + n_k} p^{n+k-(n_1 + \cdots + n_k)}, \quad n \geq 0.$$

But the nonnegative integers $n_1, \ldots, n_k$ may vary, subject to the condition $n_1 + 2n_2 + \cdots + kn_k = n$. Consequently,

$$
\begin{aligned}
P[N_k &= n + k] \\
&= P\left[\text{all } x_1 x_2 \ldots x_{n_1+\cdots+n_k} \underbrace{ss \ldots s}_{k};\ n_j \geq 0\ (1 \leq j \leq n),\ \sum_{j=1}^{k} jn_j = n\right] \\
&= \sum_{n_1, \ldots, n_k} \binom{n_1 + \cdots + n_k}{n_1, \ldots, n_k} (1 - p)^{n_1+\cdots+n_k} p^{n+k-(n_1+\cdots+n_k)},\ n \geq 0,
\end{aligned}
$$

where the summation is over all $n_1, \ldots, n_k$ as above, and this establishes the theorem.

We now have the following obvious corollary to the theorem.

COROLLARY 3.1: Let N_k be as in Theorem 3.1, and assume $k = 2$. Then

$$P[N_2 = n + 2] = \sum_{i=0}^{[n/2]} \binom{n-i}{i} p^{i+2}(1 - p)^{n-i},\ n \geq 0.$$

This result is a simpler version of Turner's [7] formula for general p and $k = 2$ (our notation).

We also have the following corollary, by means of Theorem 2.1.

COROLLARY 3.2: Let N_k be as in Theorem 3.1, and assume $p = 1/2$. Then

$$P[N_k = n + k] = f_{n+1}^{(k)}/2^{n+k},\ n \geq 0.$$

This result is a version of formula (12) of Shane [6] and of Turner's [7] formula for general k and $p = 1/2$.

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