Therefore, only the Fibonacci and Lucas sequences, and (real) multiples of them, satisfy our requirement that the right-hand side of (2) reduce to a *single* term.

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# COMPOSITION ARRAYS GENERATED BY FIBONACCI NUMBERS


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The number of compositions of an integer $n$ in terms of ones and twos [1] is $F_{n+1}$, the $(n + 1)$st Fibonacci number, defined by

$$F_0 = 0,\ F_1 = 1,\ \text{and}\ F_{n+2} = F_{n+1} + F_n.$$

Further, the Fibonacci numbers can be used to generate such composition arrays [2], leading to the sequences $A = \{a_n\}$ and $B = \{b_n\}$, where $(a_n, b_n)$ is a safe pair in Wythoff's game [3], [4], [6].

We generalize to the Tribonacci numbers $T_n$, where

$$T_0 = 0,\ T_1 = T_2 = 1,\ \text{and}\ T_{n+3} = T_{n+2} + T_{n+1} + T_n.$$

The Tribonacci numbers give the number of compositions of $n$ in terms of ones, twos, and threes [5], and when Tribonacci numbers are used to generate a composition array, we find that the sequences $A = \{A_n\}$, $B = \{B_n\}$, and $C = \{C_n\}$ arise, where $A_n$, $B_n$, and $C_n$ are the sequences studied in [7].

## 1. The Fibonacci Composition Array

To form the Fibonacci composition array, we use the difference of the subscripts of Fibonacci numbers to obtain a listing of the compositions of $n$ in terms of ones and twos, by using $F_{n+1}$ in the rightmost column, and taking the Fibonacci numbers as placeholders. We index each composition in the order in which it was written in the array by assigning each to a natural number taken in order and, further, assign the index $k$ to set $A$ if the $k$th composition has a one in the first position, and to set $B$ if the $k$th composition has a two in the first position. We illustrate for $n = 6$, using $F_7$ to write the rightmost column. Notice that every other column in the table is the subscript difference of the two adjacent Fibonacci numbers, and compare with the compositions of 6 in terms of ones and twos.

FIBONACCI SCHEME TO FORM ARRAY OF COMPOSITIONS OF INTEGERS

| | | | | | | | | | | | | | INDEX: $A$ or $B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_1$ | 1 | $F_2$ | 1 | $F_3$ | 1 | $F_4$ | 1 | $F_5$ | 1 | $F_6$ | 1 | $F_7$ | $1 = a_1$ |
| | | $F_1$ | 2 | $F_3$ | 1 | $F_4$ | 1 | $F_5$ | 1 | $F_6$ | 1 | $F_7$ | $2 = b_1$ |
| | | $F_1$ | 1 | $F_2$ | 2 | $F_4$ | 1 | $F_5$ | 1 | $F_6$ | 1 | $F_7$ | $3 = a_2$ |
| | | $F_1$ | 1 | $F_2$ | 1 | $F_3$ | 2 | $F_5$ | 1 | $F_6$ | 1 | $F_7$ | $4 = a_3$ |
| | | | | $F_1$ | 2 | $F_3$ | 2 | $F_5$ | 1 | $F_6$ | 1 | $F_7$ | $5 = b_2$ |
| | | $F_1$ | 1 | $F_2$ | 1 | $F_3$ | 1 | $F_4$ | 2 | $F_6$ | 1 | $F_7$ | $6 = a_4$ |
| | | | | $F_1$ | 2 | $F_3$ | 1 | $F_4$ | 2 | $F_6$ | 1 | $F_7$ | $7 = b_3$ |
| | | | | $F_1$ | 1 | $F_2$ | 2 | $F_4$ | 2 | $F_6$ | 1 | $F_7$ | $8 = a_5$ |
| | | $F_1$ | 1 | $F_2$ | 1 | $F_3$ | 1 | $F_4$ | 1 | $F_5$ | 2 | $F_7$ | $9 = a_6$ |
| | | | | $F_1$ | 2 | $F_3$ | 1 | $F_4$ | 1 | $F_5$ | 2 | $F_7$ | $10 = b_4$ |
| | | | | $F_1$ | 1 | $F_2$ | 2 | $F_4$ | 1 | $F_5$ | 2 | $F_7$ | $11 = a_7$ |
| | | | | $F_1$ | 1 | $F_2$ | 1 | $F_3$ | 2 | $F_5$ | 2 | $F_7$ | $12 = a_8$ |
| | | | | | | $F_1$ | 2 | $F_3$ | 2 | $F_5$ | 2 | $F_7$ | $13 = b_5$ |

One first writes the column of 13 $F_7$'s, which is broken into 8 $F_6$'s and 5 $F_5$'s. The 8 $F_6$'s are broken into 5 $F_5$'s and 3 $F_4$'s, and the 5 $F_5$'s are broken into 3 $F_4$'s and 2 $F_3$'s. The pattern continues in each column until each $F_2$ is broken into $F_1$ and $F_0$, so ending with $F_1$. In each new column, one always replaces $F_nF_n$'s with $F_{n-1}F_{n-1}$'s and $F_{n-2}F_{n-2}$'s. Note that the next level, representing all integers through $F_8 = 21$, would be formed by writing 21 $F_8$'s in the right column, and the present array as the top $13 = F_7$ rows, and the array ending in 8 $F_6$'s now in the top $8 = F_6$ rows would appear in the bottom 8 rows. Notice further that this scheme puts a one on the right of all compositions of $(n - 1)$ and a two on the right of all compositions of $(n - 2)$.

Now, we examine sets $A$ and $B$.

| $n$: | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $a_n$: | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 12 | 14 | 16 ... |
| $b_n$: | 2 | 5 | 7 | 10 | 13 | 15 | 18 | 20 | 23 | 26 ... |

Notice that $A$ is characterized as being the set of smallest integers not yet used, while it appears that $b_n = a_n + n$. Indeed, it appears that, for small values of $n$, $a_n$ and $b_n$ are the numbers arising as the safe pairs in the solution of Wythoff's game, where it is known that [2]

$$a_n = [n\alpha], \qquad b_n = [n\alpha^2], \tag{1.1}$$

where $[x]$ is the greatest integer in $x$ and $\alpha = (1 + \sqrt{5})/2$. Further, we can characterize $A$ and $B$ by

(1.2) $$a_m = 1 + \alpha_3 F_3 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\},$$

$$b_m = 2 + \alpha_4 F_4 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\}.$$

Any integer $n$ has a unique Fibonacci Zeckendorf representation

(1.3) $$n = \alpha_2 F_2 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k,$$

where $\alpha_i \in \{0, 1\}$ and $\alpha_i \alpha_{i-1} = 0$, or, a representation as a sum of distinct Fibonacci numbers where no two consecutive Fibonacci numbers may be used. Now suppose 1 is the smallest term in the Zeckendorf representation of $n$. Then $n$ is in the required form for $a_m$. Suppose that the smallest Fibonacci number used is $F_k$, where $k$ is even. Replace $F_k$ by $F_{k-1} + F_{k-2}$, $F_{k-2}$ by $F_{k-3} + F_{k-4}$, $F_{k-4}$ by $F_{k-5} + F_{k-6}$, ..., until one reaches $F_4 = F_3 + F_2$, so that we have smallest term 1, and the required form for $a_m$.

Similarly, if 2 is the smallest term in the representation of $n$, then $n$ is in the required form for $b_m$. If the subscript of the smallest Fibonacci number used is odd, then we can replace $F_k$ by $F_{k-1} + F_{k-2}$, $F_{k-2}$ by $F_{k-3} + F_{k-4}$, ..., just as before, until we reach $F_5 = F_4 + F_3$, equivalent to ending in a 2 for the form of $b_m$.

Thus $A$ is the set of numbers whose Zeckendorf representation has an even-subscripted smallest term, while elements of $B$ have odd-subscripted smallest terms. Since the Zeckendorf representation is unique, $A$ and $B$ are disjoint and cover the set of positive integers. Also, the unique Zeckendorf representation allows us to modify the form to that given for $a_m$ and $b_m$ uniquely, by rewriting only the smallest term.

Now, we can prove that $A$ and $B$ do indeed contain the safe-pair sequences from Wythoff's game.

Theorem 1.1: Form the composition array for $n$ in terms of ones and twos, using $F_{n+1}$ on the right border. Number the compositions in order appearing. Then, if 1 appears as the first number in the $k$th composition,

$$k = a_m = 1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\},$$

and if 2 appears as the first number in the $k$th composition,

$$k = b_m = 2 + \alpha_4 F_4 + \alpha_5 F_5 + \cdots + \alpha_k F_k,\ \alpha_i \in \{0, 1\},$$

where $(a_m, b_m)$ is a safe pair in Wythoff's game.

Proof: We have seen this for $n = 6$ and $k = 1, 2, \ldots, 13 = F_7$, and by using subarrays found there, we could illustrate $n = 1, 2, 3, 4$, and 5. By the construction of the array, we can build a proof by induction.

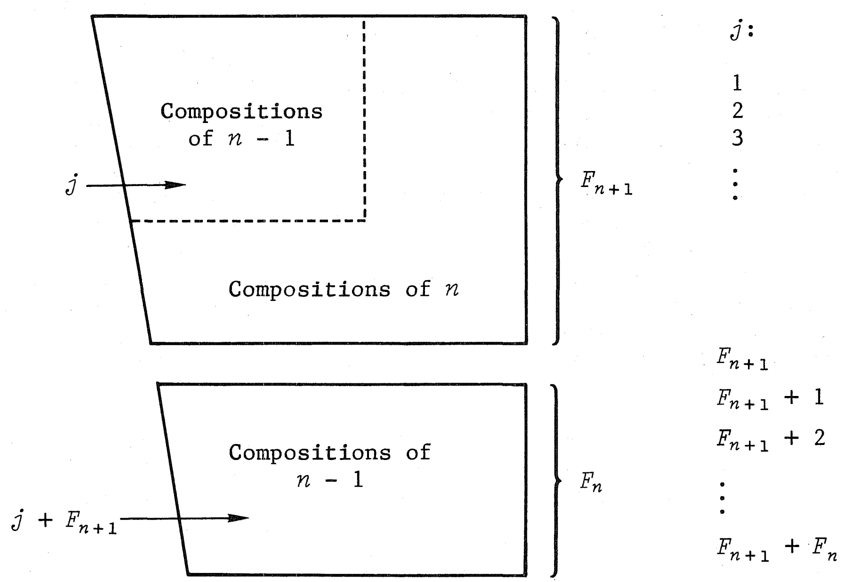
Assume we have the compositions of $n$ using ones and twos made by our construction, using $F_{n+1}$ in the rightmost column. We put the $F_n$ compositions of $(n - 1)$ below. (See figure on page 125.)

Take $1 \le j \le F_n$. If $j \in A$, then $j + F_{n+1} \in A$ as the compositions starting with 1 go into $A$ and those starting with 2 go into $B$, and addition of $F_{n+1}$ will not affect earlier terms used. Note well that no matter how large the value of $n$ becomes, the earlier compositions always start with the same number 1 or 2 as they did for the smaller value of $n$, within the range of the construction. Now, if $j$ is of the form

$$j = 1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_n F_n,\ j \in A,$$

$$j + F_{n+1} = 1 + \alpha_3 F_3 + \alpha_4 F_4 + \cdots + \alpha_n F_n + F_{n+1},$$

and $(j + F_{n+1}) \in A$, $\alpha_i \in \{0, 1\}$. Since we know that all the integers from 1 to $F_{n+2} - 2$ can be represented with the Fibonacci numbers 1, 2, 3, ..., $F_n$, for the numbers through $F_{n+1}$ we need only $F_2, F_3, \ldots, F_n$. Thus the numbers 1, 2, ..., $F_{n+2}$ can be represented using 1, 2, ..., $F_{n+1}$, and we continue to build the sets $A$ and $B$, having both completeness and uniqueness, recalling [1] that the number of compositions of $n$ into ones and twos is $F_{n+1}$. Also, notice that there are $F_{n-2}$ elements of $B$ in the first $F_n$ integers and $F_{n-1}$ elements of $A$ in the first $F_n$ integers.




$1 \leq j \leq F_n$, $n \geq 2$.

## 2. The Tribonacci Composition Array

Normally, the Tribonacci numbers give rise to three sets $A$, $B$, $C$ [8]:

$$\begin{aligned} A &= \{A_n : A_n = 1 + \alpha_3 T_3 + \alpha_4 T_4 + \cdots\}, \\ B &= \{B_n : B_n = 2 + \alpha_4 T_4 + \alpha_5 T_5 + \cdots\}, \\ C &= \{C_n : C_n = 4 + \alpha_5 T_5 + \alpha_6 T_6 + \cdots\}, \end{aligned} \tag{2.1}$$

where $\alpha_i \in \{0, 1\}$. Equivalently, see [7], if $T_k$ is the smallest term appearing in the unique Zeckendorf representation of an integer $N$, then

$N \in A$ if $k \equiv 2 \bmod 3$, $N \in B$ if $k \equiv 3 \bmod 3$, and $N \in C$ if $k \equiv 1 \bmod 3$, $k > 3$,

where we have suppressed $T_1 = 1$, but $T_2 = 1 = A_1$, and every positive integer belongs to $A$, $B$, or $C$, where $A$, $B$, and $C$ are disjoint.

Also, recall that the compositions of a positive integer $n$ using 1's, 2's, and 3's gives rise to the Tribonacci numbers, since $T_{n+1}$ gives the number of such compositions [5].

Now, proceeding as in the Fibonacci case, we write a Tribonacci composition array. We illustrate for $T_6 = 13$ in the rightmost column, which is the number of compositions of 5 into 1's, 2's, and 3's. We put the index of those compositions which start on the left with a one into set $A$, those with a two into set $B$, and those with a three into set $C$, and compare with sets $A$, $B$, and $C$ given in (2.1).

TRIBONACCI SCHEME TO FORM ARRAY OF COMPOSITIONS OF INTEGERS

| | | | | | | | | | | | INDEX: $A$, $B$, or $C$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $T_1$ | 1 | $T_2$ | 1 | $T_3$ | 1 | $T_4$ | 1 | $T_5$ | 1 | $T_6$ | $1 = A_1$ |
| | | $T_1$ | 2 | $T_3$ | 1 | $T_4$ | 1 | $T_5$ | 1 | $T_6$ | $2 = B_1$ |
| | | $T_1$ | 1 | $T_2$ | 2 | $T_4$ | 1 | $T_5$ | 1 | $T_6$ | $3 = A_2$ |
| | | | | $T_1$ | 3 | $T_4$ | 1 | $T_5$ | 1 | $T_6$ | $4 = C_1$ |
| | | $T_1$ | 1 | $T_2$ | 1 | $T_3$ | 2 | $T_5$ | 1 | $T_6$ | $5 = A_3$ |
| | | | | $T_1$ | 2 | $T_3$ | 2 | $T_5$ | 1 | $T_6$ | $6 = B_2$ |
| | | | | $T_1$ | 1 | $T_2$ | 3 | $T_5$ | 1 | $T_6$ | $7 = A_4$ |
| | | $T_1$ | 1 | $T_2$ | 1 | $T_3$ | 1 | $T_4$ | 2 | $T_6$ | $8 = A_5$ |
| | | | | $T_1$ | 2 | $T_3$ | 1 | $T_4$ | 2 | $T_6$ | $9 = B_3$ |
| | | | | $T_1$ | 1 | $T_2$ | 2 | $T_4$ | 2 | $T_6$ | $10 = A_6$ |
| | | | | | | $T_1$ | 3 | $T_4$ | 2 | $T_6$ | $11 = C_2$ |
| | | | | $T_1$ | 1 | $T_2$ | 1 | $T_3$ | 3 | $T_6$ | $12 = A_7$ |
| | | | | | | $T_1$ | 2 | $T_3$ | 3 | $T_6$ | $13 = B_4$ |

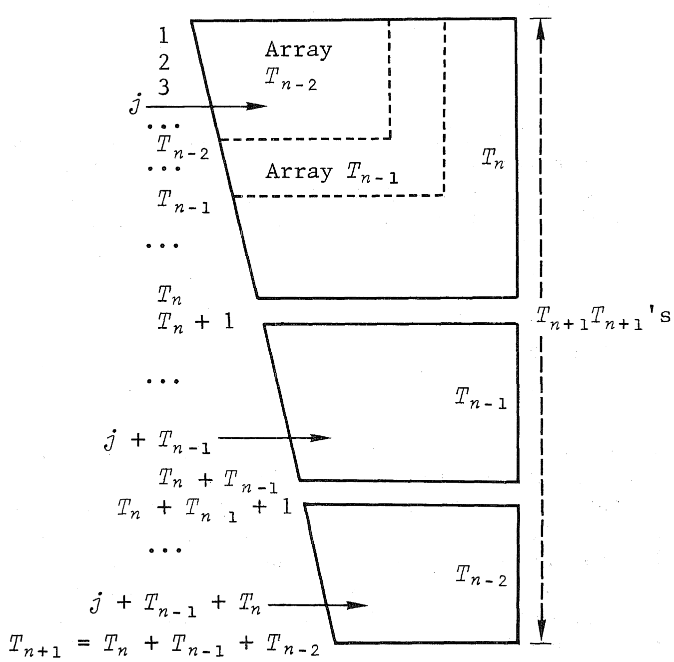
We note that thus far the splitting into sets agrees with these rules: $A_n$ is the first positive integer not yet used; $B_n$ is $2A_n$ decreased by the number of $C_i$'s less than $A_n$; and $C_n$ is $2B_n$ decreased by the number of $C_i$'s less than $B_n$; where $A_n$, $B_n$, and $C_n$ are the elements of sets $A$, $B$, and $C$ of (2.1).

| $n$: | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $A_n$: | 1 | 3 | 5 | 7 | 8 | 10 | 12 |
| $B_n$: | 2 | 6 | 9 | 13 | 15 | 19 | 22 |
| $C_n$: | 4 | 11 | 17 | 24 | 28 | 35 | 41 |

We next prove that this constructive array yields the same sets $A, B$, and $C$ as characterized by (2.1) by mathematical induction.

We first study the array we have written, yielding the $T_6 = 13$ compositions of $n = 5$ using 1's, 2's, and 3's. We write 13 $T_6$'s in the rightmost column. Then, write the preceding column on the left by dividing 13 $T_6$'s into 7 $T_5$'s, 4 $T_4$'s, and 2 $T_3$'s. In successive columns, replace the 7 $T_5$'s by 4 $T_4$'s, 2 $T_3$'s, and 1 $T_2$, and the 4 $T_4$'s by 2 $T_3$'s, 1 $T_2$, and 1 $T_1$, then the 2 $T_3$'s by 1 $T_2$ and 1 $T_1$. Any row that reaches $T_1$ stops. Continue until all the rows have reached $T_1$. Notice that the top left corner of the array, bordered by 7 $T_5$'s on the right, is the array for the $T_5 = 7$ compositions of $n - 1 = 4$, and that the middle group bordered on the right by 4 $T_4$'s is the $4 = T_4$ compositions of $n - 2 = 3$, and the bottom group bordered by 2 $T_3$'s on the right is the $2 = T_3$ compositions of $n - 3 = 2$. The successive subscript differences give the compositions of $n$ using 1's, 2's, and 3's.

If we write $T_{n+1}T_{n+1}$'s in the right-hand column, then we will have in the preceding column the arrays formed from $T_nT_n$'s on the right, $T_{n-1}T_{n-1}$'s on the right, and $T_{n-2}T_{n-2}$'s on the right. All the integers from 1 through $T_{n+1}$ will appear as indices because there are $T_{n+1}$ compositions of $n$ into 1's, 2's, and 3's. The subscript differences will give the compositions of $n$ into 1's, 2's, and 3's, and we can make a correspondence between the natural numbers, the compositions of $n$, and the representative form of the appropriate set. Those $T_{n+1}$ compositions are ordered with indices from the natural numbers. Each composition whose leftmost digit is one is cast into set $A$; those whose leftmost digit is two are cast into set $B$; and those whose leftmost digit is three are cast into set $C$. Descending the list, we then call the first $A$, $A_1$, the second $A$, $A_2$, ..., the first $B$, $B_1$, the second $B$, $B_2$, and so on. We have now listed the elements of $A$, $B$, and $C$ in natural order. Since the representations of $A_n$, $B_n$, and $C_n$ from (2.1) are unique, see [7], and since this expansion is constructively derived from the Zeckendorf representation so that the largest term used remains intact (by the lexicographic ordering theorem [7]), every integer $m < T_n$ uses only $T_2, T_3, \ldots, T_{n-1}$ in the representation, and $T_n$ can itself be written such that the largest term used is $T_{n-1}$. Let $j$ by any integer, $1 \leq j \leq T_n$. Assume that $j$ can be expressed as in (2.1). Then $j' = j + T_{n+1}$ will be in the same set as $j$, since all early terms of $j$ and $j'$ will be the same. Further, if the leftmost digit of the $j$th composition is $a$, where $1 \leq j \leq T_n$, then the leftmost digit of the $j'$th composition, $j' = j + T_{n+1}$ will be $a$, since the leftmost digits are not changed in construction of the array.



Recall that set elements are not characterized by the composition, but only by its leading 1, 2, or 3. Each number in the $j$th position in the original gives rise to one in the $(j + T_{n+1})$st position in the same set $A$, $B$, or $C$. Also, $j + T_n + T_{n+1}$ belongs to the same set as $j$.

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