PYTHAGOREAN TRIPLES

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(Submitted June 1980)

Define the sequence $\{w_n\}$ by

$$w_0 = a,\ w_1 = b,\ w_{n+2} = w_{n+1} + w_n \tag{1}$$

(n integer ≥ 0; a, b real and both not zero).

Then [2],

$$(w_n w_{n+3})^2 + (2w_{n+1}w_{n+2})^2 = (w_{n+1}^2 + w_{n+2}^2)^2. \tag{2}$$

Freitag [1] asks us to find a c_n, if it exists, for which

$$(F_n F_{n+3},\ 2F_{n+1}F_{n+2},\ c_n) \tag{3}$$

is a Pythagorean triple, where F_n is the nth Fibonacci number. It is easy to show that $c_n = F_{2n+3}$.

Earlier, Wulczyn [3] had shown that

$$(L_n L_{n+3},\ 2L_{n+1}L_{n+2},\ 5F_{2n+3}) \tag{4}$$

is a Pythagorean triple, where L_n is the nth Lucas number.

Clearly, (3) and (4) are special cases of (2) in which $a = 0$, $b = 1$, and $a = 2$, $b = 1$, respectively. One would like to know whether (3) and (4) provide the only solutions of (2) in which the third element of the triple is a *single* term. Our feeling is that they do.

Now

$$w_n = aF_{n-1} + bF_n, \tag{5}$$

so

$$w_{n+1}^2 + w_{n+2}^2 = \begin{cases} (a^2 + b^2)F_{2n+3} + (2ab - a^2)F_{2n+2} \\ (b^2 + 2ab)F_{2n+3} + (a^2 - 2ab)F_{2n+1} \end{cases} \tag{6}$$

$$= \begin{cases} \left.\begin{array}{ll} b^2F_{2n+3} & \text{if } a = 0 \\ 5b^2F_{2n+3} & \text{if } a = 2b \end{array}\right\} \mathrm{I} \\ \left.\begin{array}{ll} a^2F_{2n+1} & \text{if } b = 0 \\ 5a^2F_{2n+1} & \text{if } b = -2a \end{array}\right\} \mathrm{II} \end{cases} \tag{7}$$

whence

$$w_n = \begin{cases} bF_n \quad \text{or} \quad bL_n \quad \text{by I} \\ aF_{n-1} \quad \text{or} \quad -aL_{n-1} \quad \text{by II,} \end{cases} \tag{8}$$

results which may be verified in (5).

Therefore, only the Fibonacci and Lucas sequences, and (real) multiples of them, satisfy our requirement that the right-hand side of (2) reduce to a *single* term.

References

1. H. Freitag. Problem B-426. *The Fibonacci Quarterly* 18, no. 2 (1980):186.
2. A. F. Horadam. "Special Properties of the Sequence $w_n(a, b; p, q)$." *The Fibonacci Quarterly* 5, no. 5 (1967):424-434.
3. G. Wulczyn. Problem B-402. *The Fibonacci Quarterly* 18, no. 2 (1980):188.

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COMPOSITION ARRAYS GENERATED BY FIBONACCI NUMBERS

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(Submitted September 1980)

The number of compositions of an integer n in terms of ones and twos [1] is F_{n+1}, the $(n + 1)$st Fibonacci number, defined by

$$F_0 = 0,\ F_1 = 1, \text{ and } F_{n+2} = F_{n+1} + F_n.$$

Further, the Fibonacci numbers can be used to generate such composition arrays [2], leading to the sequences $A = \{a_n\}$ and $B = \{b_n\}$, where (a_n, b_n) is a safe pair in Wythoff's game [3], [4], [6].

We generalize to the Tribonacci numbers T_n, where

$$T_0 = 0,\ T_1 = T_2 = 1, \text{ and } T_{n+3} = T_{n+2} + T_{n+1} + T_n.$$

The Tribonacci numbers give the number of compositions of n in terms of ones, twos, and threes [5], and when Tribonacci numbers are used to generate a composition array, we find that the sequences $A = \{A_n\}$, $B = \{B_n\}$, and $C = \{C_n\}$ arise, where A_n, B_n, and C_n are the sequences studied in [7].

1. The Fibonacci Composition Array

To form the Fibonacci composition array, we use the difference of the subscripts of Fibonacci numbers to obtain a listing of the compositions of n in terms of ones and twos, by using F_{n+1} in the rightmost column, and taking the Fibonacci numbers as placeholders. We index each composition in the order in which it was written in the array by assigning each to a natural number taken in order and, further, assign the index k to set A if the kth composition has a one in the first position, and to set B if the kth composition has a two in the first position. We illustrate for $n = 6$, using F_7 to write the rightmost