BINET'S FORMULA FOR THE TRIBONACCI SEQUENCE

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1. Introduction

The terms of a recursive sequence are usually defined by a recurrence procedure; that is, any term is the sum of preceding terms. Such a definition might not be entirely satisfactory, because the computation of any term could require the computation of all of its predecessors. An alternative definition gives any term of a recursive sequence as a function of the index of the term. For the simplest nontrivial recursive sequence, the Fibonacci sequence, Binet's formula [1]

$$u_n = (1/\sqrt{5})(\alpha^{n+1} - \beta^{n+1})$$

defines any Fibonacci number as a function of its index and the constants

$$\alpha = \frac{1}{2}(1 + \sqrt{5}) \quad \text{and} \quad \beta = \frac{1}{2}(1 - \sqrt{5}).$$

In this paper, an analog of Binet's formula for the Tribonacci sequence

$$1,\ 1,\ 2,\ 4,\ 7,\ \ldots,\ u_{n+1} = u_n + u_{n-1} + u_{n-2},\ \ldots$$

(see [2]), is derived. Binet's formula defines any term of the Tribonacci sequence as a function of the index of the term and three constants, ρ, σ, and τ.

2. Binet's Formula for the Tribonacci Sequence

Binet's formula is derived by determining the generating function for the difference equation

$$u_0 = u_1 = 1,\ u_2 = 2$$

$$u_{n+1} = u_n + u_{n-1} + u_{n-2} \qquad n \geq 2.$$

Let $f(x) = u_0 + u_1 x + u_2 x^2 + \cdots + u_n x^n + \cdots = \sum_{i=0}^{\infty} u_i x^i$ be the generating function; then

$$(1 - x - x^2 - x^3) f(x) = 1,$$

so

$$f(x) = \frac{1}{1 - x - x^2 - x^3} = \frac{1}{(1 - \rho x)(1 - \sigma x)(1 - \tau x)} = \frac{1}{p(x)}.$$

The roots of $p(x) = 0$ are $1/\rho$, $1/\sigma$, and $1/\tau$, where ρ, σ, and τ are the roots of

$$p\left(\frac{1}{x}\right) = x^3 - x^2 - x - 1 = 0.$$

Applying Cardan's formulas to $p\left(\frac{1}{x}\right) = 0$ yields

$$\rho = \frac{1}{3}\left(\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1\right),$$

$$\sigma = \frac{1}{6}\left(\left[2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}} + \sqrt{3}i\ \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}\right]\right),$$

and

$\tau = \overline{\sigma}$, the complex conjugate of σ.

Approximate numerical values for ρ, σ, and $\overline{\sigma}$ are:

$$\rho = 1.8393,\ \sigma = -0.4196 + 0.6063i,\ \overline{\sigma} = -0.4196 - 0.6063i.$$

Since the roots of $p(x) = 0$ are distinct, by partial fractions

$$f(x) = \frac{1}{(1 - \rho x)(1 - \sigma x)(1 - \overline{\sigma}x)} = \frac{A}{1 - \rho x} + \frac{B}{1 - \sigma x} + \frac{C}{1 - \overline{\sigma}x}.$$

Here

$$A = \frac{1}{\left(1 - \frac{\sigma}{\rho}\right)\left(1 - \frac{\overline{\sigma}}{\rho}\right)} = \frac{\rho^2}{(\rho - \sigma)(\rho - \overline{\sigma})},$$

$$B = \frac{1}{\left(1 - \frac{\rho}{\sigma}\right)\left(1 - \frac{\overline{\sigma}}{\sigma}\right)} = \frac{\sigma^2}{(\sigma - \rho)(\sigma - \overline{\sigma})},$$

and

$$C = \frac{1}{\left(1 - \frac{\rho}{\overline{\sigma}}\right)\left(1 - \frac{\sigma}{\overline{\sigma}}\right)} = \frac{\overline{\sigma}^2}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}.$$

Consequently,

$$f(x) = \frac{\rho^2}{(\rho - \sigma)(\rho - \overline{\sigma})}\sum_{i=0}^{\infty}\rho^i x^i + \frac{\sigma^2}{(\sigma - \rho)(\sigma - \overline{\sigma})}\sum_{i=0}^{\infty}\sigma^i x^i + \frac{\overline{\sigma}^2}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}\sum_{i=0}^{\infty}\overline{\sigma}^i x^i$$

$$= \sum_{i=0}^{\infty}\left(\frac{\rho^{i+2}}{(\rho - \sigma)(\rho - \overline{\sigma})} + \frac{\sigma^{i+2}}{(\sigma - \rho)(\sigma - \overline{\sigma})} + \frac{\overline{\sigma}^{i+2}}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}\right)x^i.$$

Thus, Binet's formula for the Tribonacci sequence is

$$u_n = \frac{\rho^{n+2}}{(\rho - \sigma)(\rho - \overline{\sigma})} + \frac{\sigma^{n+2}}{(\sigma - \rho)(\sigma - \overline{\sigma})} + \frac{\overline{\sigma}^{n+2}}{(\overline{\sigma} - \rho)(\overline{\sigma} - \sigma)}.$$

Multiplying the numerators and denominators of the last two terms by $(\rho - \bar{\sigma})$ and $(\rho - \sigma)$, respectively, yields

$$u_n = \frac{\rho^{n+2}}{|\rho - \sigma|^2} + \frac{(\rho - \bar{\sigma})\sigma^{n+2}}{-2iI(\sigma)|\rho - \sigma|^2} + \frac{(\rho - \sigma)\bar{\sigma}^{n+2}}{2iI(\sigma)|\rho - \sigma|^2}.$$

Using the relations $\sigma = r(\cos\theta + i\sin\theta)$,

$$\sigma^n = r^n(\cos n\theta + i\sin n\theta),\ \theta = \tan^{-1}(I(\sigma)/R(\sigma))$$

and combining terms:

$$u_n = \frac{\rho^2}{|\rho - \sigma|^2}\rho^n + \frac{r(r - 2\rho\cos\theta)}{|\rho - \sigma|^2} r^n \cos n\theta$$
$$+ \frac{r^2\cos\theta - \rho r(1 - 2\sin^2\theta)}{\sin\theta|\rho - \sigma|^2} r^n \sin n\theta.$$

Denoting the coefficients of ρ^n, $r^n\cos n\theta$, and $r^n\sin n\theta$ by α, β, and γ, respectively, yields

$$u_n = \alpha\rho^n + r^n(\beta\cos n\theta + \gamma\sin n\theta).$$

Approximate values for the constants are:

$$\rho = 1.8393, \quad \theta = 124.69°, \quad r = 0.7374,$$
$$\alpha = 0.6184, \quad \beta = 0.3816, \quad \gamma = 0.0374.$$

3. An Application

Since $|r| = .7374 < 1$, the nth Tribonacci number is the integer nearest $\alpha\rho^n$ when

$$|r^n(\beta\cos n\theta + \gamma\sin n\theta)| < \frac{1}{2}.$$

Using calculus, the value of $|\beta\cos n\theta + \gamma\sin n\theta|$ is at a maximum when

$$n\theta = 5.60° + k\pi, \text{ for } k \text{ an integer.}$$

Consequently,

$$|r^n(\beta\cos n\theta + \gamma\sin n\theta| < \frac{1}{2} \text{ for } n \geq 1.$$

Since $[\alpha + .5] = 1$ (where [ ] is the greatest integer function), a short form of the formula that is suitable for calculating the terms of the Tribonacci sequence is

$$u_n = [\alpha\rho^n + .5] \text{ for } n \geq 0.$$

References

1. Vorob'ev, N. *The Fibonacci Numbers*. Boston: Heath, 1963, pp. 12-15.
2. Feinberg, Mark. "Fibonacci-Tribonacci." *The Fibonacci Quarterly* 1, no. 1 (1963):71-74.

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