A PROPERTY OF BINOMIAL COEFFICIENTS

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The purpose of this paper is to prove identity (1), related to the binomial coefficients.

For each pair of integers $n, m \geq 0$, the following identity holds:

$$\sum_{i=0}^{m} \frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{h=0}^{n} \frac{\binom{n+m-h}{m}}{2^{n+m-h}} = 2. \tag{1}$$

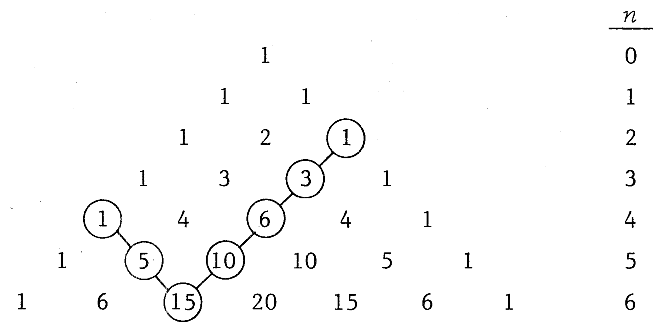
The meaning of this identity becomes more clear if one considers Pascal's triangle:

														n
						1								0
					1		1							1
				1		2		1						2
			1		3		3		1					3
		1		4		6		4		1				4
	1		5		10		10		5		1			5
1		6		15		20		15		6		1		6

Let us consider a path which starts from any point on the left side and goes down following a line parallel to the right side, then stops at any point and goes up again, following a line parallel to the left side, until it reaches the right side. If we add all the binomial coefficients we have met, each multiplied by 2^{-n}, the result is always 2. (The binomial coefficient at the turning point of the path being considered twice.) For example, the following path yields

$$\frac{\binom{4}{0}}{2^4} + \frac{\binom{5}{1}}{2^5} + \frac{\binom{6}{2}}{2^6} + \frac{\binom{6}{2}}{2^6} + \frac{\binom{5}{2}}{2^5} + \frac{\binom{4}{2}}{2^4} + \frac{\binom{3}{2}}{2^3} + \frac{\binom{2}{2}}{2^2} = 2.$$

The Pascal triangle is shown in the following figure.



To prove identity (1), we need:

Lemma 1

Let $a, b, c \in \mathbb{Z}$ with $a < b$ and $c \geq 0$. Then we have

$$2\sum_{k=a}^{b}\binom{c}{k} = \sum_{k=a+1}^{b}\binom{c+1}{k} + \binom{c}{a} + \binom{c}{b}. \tag{2}$$

Proof: This identity stems immediately from the fact that

$$\binom{c}{k} + \binom{c}{k+1} = \binom{c+1}{k+1}.$$

Now we can prove identity (1). This identity is true if $n = m = 0$. Let us assume that n is different from zero and change the index h to $j = n - h$. We obtain the following equivalent identity:

$$\sum_{i=0}^{m}\frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{j=0}^{n}\frac{\binom{m+j}{m}}{2^{m+j}} = 2, \tag{3}$$

which is symmetrical in n and m since $\binom{m+j}{m} = \binom{m+j}{j}$. We can thus assume that $m \leq n$ (and $n \neq 0$).

Let us now write the binomial theorem $(1+1)^m = 2^m = \sum_{k=0}^{m}\binom{m}{k}$ in the following form:

$$2 = \frac{\sum_{k=0}^{m}\binom{m}{k}}{2^{m-1}} = 2\frac{\sum_{k=0}^{m}\binom{m}{k}}{2^m}.$$

Since, for $k < 0$, $\binom{m}{k} = 0$, the sum may start at $k = m - n \leq 0$:

$$2 = 2\frac{\sum_{k=m-n}^{m}\binom{m}{k}}{2^m}.$$

Since $m > m - n$, applying Lemma 1 to the sum we get:

$$\frac{2\sum_{k=m-n}^{m}\binom{m}{k}}{2^m} = \frac{\sum_{k=m-n+1}^{m}\binom{m+1}{k} + \binom{m}{m-n} + \binom{m}{m}}{2^m}$$

$$= \frac{2\sum_{k=m-n+1}^{m}\binom{m+1}{k}}{2^{m+1}} + \frac{\binom{m}{m-n} + \binom{m}{m}}{2^m}.$$

Again we have $m \geq m - n + 1$, and again we can apply Lemma 1. If we proceed in this way, after using the lemma r times we get:

$$\frac{2\sum_{k=m-n+r}^{m}\binom{m+r}{k}}{2^{m+r}} \sum_{j=0}^{r-1} \frac{\binom{m+j}{m+n+j} + \binom{m+j}{m}}{2^{m+j}}.$$

So we can apply Lemma 1 until $m - n + r < m$, i.e., until $r < n$. At this point we get:

$$\frac{2\binom{m+n}{m}}{2^{m+n}} + \sum_{j=0}^{n-1} \frac{\binom{m+j}{m-n+j} + \binom{m+j}{m}}{2^{m+j}} = \sum_{j=0}^{n} \frac{\binom{m+j}{m-n+j} + \binom{m+j}{m}}{2^{m+j}}$$

$$= \sum_{s=0}^{n} \frac{\binom{m+s}{m-n+s}}{2^{m+s}} + \sum_{j=0}^{h} \frac{\binom{m+j}{m}}{2^{m+j}}.$$

If we select the index transformation $i = s + m - n$ and observe that, due to the fact that $m - n < 0$, we can restrict the range of i to nonnegative values, we obtain

$$2 = \sum_{i=0}^{m} \frac{\binom{n+i}{i}}{2^{n+i}} + \sum_{j=0}^{h} \frac{\binom{m+j}{m}}{2^{m+j}},$$

which is what we desired.

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