ROOTS OF RECURRENCE-GENERATED POLYNOMIALS

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1. Introduction

The object of this note is to synthesize information relating to certain polynomials forming the subject matter of [1], [2], [3], and [4], the notation of which will be used hereafter. In the process, a verification of the roots of the Fibonacci and Lucas polynomials obtained in [2] is effected.

Polynomials $A_n(x)$ were defined in [3] by

$$\begin{cases} A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1 \text{ and} \\ A_n(x) = xA_{n-2}(x) - A_{n-4}(x). \end{cases} \tag{1.1}$$

Squares of the roots of

$$\frac{A_{4n}(x)}{x} = 0 \tag{1.2}$$

(of degree $2n - 2$), associated with the Chebyshev polynomial of the second kind, $U_n(x)$, were shown in [4] to be given by

$$4\cos^2\frac{i\pi}{2n} \qquad (i = 1, 2, \ldots, n - 1). \tag{1.3}$$

The actual roots may be written

$$\pm 2\sin\frac{(n - i)\pi}{2n} \qquad (i = 1, 2, \ldots, n - 1) \tag{1.4}$$

or, what amounts to the same thing,

$$\pm 2\sin\frac{i\pi}{2n} \qquad (i = 1, 2, \ldots, n - 1). \tag{1.5}$$

Proper divisors were defined in [4] as follows: "For any sequence $\{u_n\}$, $n \geq 1$, where $u_n \in \mathbf{Z}$ or $u_n(x) \in \mathbf{Z}(x)$, the *proper divisor* w_n is the quantity implicitly defined, for $n \geq 1$, by $w_1 = u_1$ and $w_n = \max\{d\colon d | u_n$ and g.c.d. $(d, w_m) = 1$ for every $m < n\}$."

For $\{A_n(x)\}$, the first few proper divisors are:

$w_1(x) = 1$, $w_2(x) = 1$, $w_3(x) = x + 1$, $w_4(x) = x$, $w_5(x) = x^2 + x - 1$,

$w_6(x) = x - 1$, $w_7(x) = x^3 + x^2 - 2x - 1$, $w_8(x) = x^2 - 2$,

$w_9(x) = x^3 - 3x + 1$, $w_{10}(x) = x^2 - x - 1$,

$w_{11}(x) = x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$, $w_{12}(x) = x^2 - 3$.

From the definition of proper divisors, we obtain (see [3])

$$A_n(x) = \prod_{d|n} w_d(x). \tag{1.6}$$

2. Complex Fibonacci and Lucas Polynomials

Hoggatt and Bicknell [2] defined the Fibonacci polynomials $F_n(x)$ by

$$F_1(x) = 1,\ F_2(x) = x,\ F_{n+1}(x) = xF_n(x) + F_{n-1}(x) \tag{2.1}$$

and the Lucas polynomials $L_n(x)$ by

$$L_1(x) = x,\ L_2(x) = x^2 + 2,\ L_{n+1}(x) = xL_n(x) + L_{n-1}(x). \tag{2.2}$$

Table 1 in [2] sets out the Lucas polynomials for the values $n = 1, 2, \ldots, 9$ (while Table 2 of [7] gives the coefficients of the Lucas polynomials as far as $L_{11}(x)$). Using hyperbolic functions, Hoggatt and Bicknell ([2, p. 273]) then established complex solutions of the equations

$$F_{2n}(x) = 0,\ F_{2n+1}(x) = 0,\ L_{2n}(x) = 0, \text{ and } L_{2n+1}(x) = 0,$$

which are of degree $2n - 1$, $2n$, $2n$, and $2n + 1$, respectively.

Suppose we now replace x by ix $(i = \sqrt{-1})$ in (2.1) and (2.2). Designating the new polynomials by $F_n^*(x)$ and $L_n^*(x)$, we have, from $F_{2n}(x) = F_n(x)L_n(x)$:

$$F_{2n}^*(x) = F_n^*(x)L_n^*(x). \tag{2.3}$$

Referring to the details of Table 1 in [2], we can tabulate the ensuing information where, for visual ease, we have represented the polynomials $A_n(x)$ and the proper divisors $w_n(x)$ of $A_n(x)$ by A_n and w_n, respectively (see Table 1, p. 221).

Summarizing the tabulated data, we have

$$F_{2n}^*(x) = (-1)^{n-1} iA_{4n}(x) \qquad (n \geq 1), \tag{2.4}$$

$$F_{2n+1}^*(x) = (-1)^n A_{4n+2}(x) = (-1)^n \Psi_{2n}(x) \qquad (n \geq 0), \tag{2.5}$$

$$L_{2n}^*(x) = (-1)^n B_{8n}(x) \qquad (n \geq 1), \tag{2.6}$$

$$L_{2n+1}^*(x) = (-1)^n ixB_{4(2n+1)}(x) = (-1)^n ix\Phi_{2n}(x) \qquad (n \geq 1), \tag{2.7}$$

TABLE 1

n	$F_n^*(x)$ n even	$F_n^*(x)$ n odd	$L_n^*(x)$ n even	$L_n^*(x)$ n odd
1		A_2		ixw_1
2	iA_4		$-w_8$	
3		$-A_6$		$-ixw_{12}$
4	$-iA_8$		w_{16}	
5		A_{10}		ixw_{20}
6	iA_{12}		$-w_8w_{24}$	
7		$-A_{14}$		$-ixw_{28}$
8	$-iA_{16}$		w_{32}	
9		A_{18}		$ixw_{12}w_{36}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

where the symbolism $\Psi_{2n}(x)$ and $\Phi_{2n}(x)$ of Hancock [1] has been introduced in (2.5) and (2.7). For the $B_{4n}(x)$ notation given in terms of proper divisors, see [4, p. 248]. Degrees of $F_n^*(x)$ and $L_n^*(x)$ are, of course, the same as those of the corresponding $F_n(x)$ and $L_n(x)$.

The results of (2.4) and (2.5) follow directly from (1.4) of [4] and the well-known fact:

$$F_n(x) = \sum_{j=0}^{[(n-1)/2]} \binom{n-j-1}{j} x^{n-2j-1}.$$

To establish (2.6) and (2.7), we consider the evenness and oddness of n separately and invoke (4.1) of [4].

<u>n even</u> $(n = 2k)$: $L_{2k}^*(x) = \dfrac{F_{4k}^*(x)}{F_{2k}^*(x)}$ by (2.3)

$$= \frac{(-1)^{2k-1} iA_{8k}(x)}{(-1)^{k-1} iA_{4k}(x)} \quad \text{by (2.4)}$$

$$= (-1)^k \frac{A_{8k}(x)}{A_{4k}(x)} = (-1)^k B_{8k}(x) \quad \text{by (4.1) of [4].}$$

<u>n odd</u> $(n = 2k + 1)$: $L_{2k+1}^*(x) = \dfrac{F_{4k+2}^*(x)}{F_{2k+1}^*(x)}$ by (2.3)

$$= \frac{(-1)^{2k} iA_{8k+4}(x)}{(-1)^k A_{4k+2}(x)} \quad \text{by (2.4), (2.5)}$$

$$= (-1)^k ixB_{8k+4}(x) \quad \text{by (4.1) of [4].}$$

From (2.6) and (2.7), an explicit formula for $B_{4n}(x)$ may be obtained by appealing to the known expression for $L_n(x)$:

$$L_n(x) = \sum_{j=0}^{[n/2]} \frac{n}{n-j}\binom{n-j}{j}x^{n-2j} \qquad (n \geq 1).$$

Arguing for $A_{4n+2}(2x) = U_{2n}(x)$ (the Chebyshev polynomial of the second kind) as for $A_{4n}(x)$ in [4], we derive the $2n$ roots of

$$A_{4n+2}(x) = 0 \tag{2.8}$$

to be $\pm 2 \cos \dfrac{i\pi}{2n+1}$ $(i = 1, 2, \ldots, n)$ or, equivalently,

$$\pm 2 \sin \frac{(2i+1)}{(2n+1)} \cdot \frac{\pi}{2} \qquad (i = 0, 1, 2, \ldots, n-1). \tag{2.9}$$

Next, consider the roots of

$$B_{8n}(x) = 0. \tag{2.10}$$

From [4], these are the roots of $\dfrac{A_{8n}(x)}{x} = 0$ excluding those belonging to the set of roots of (1.2). Consequently, by (1.4), the roots of (2.10) are $\pm 2 \sin \dfrac{(2n-i)\pi}{4n}$ $(i = 1, 2, \ldots, 2n-1)$ diminished by $\pm 2 \sin \dfrac{2(n-i)\pi}{4n}$ $(i = 1, 2, \ldots, n-1)$. Calculation yields the remaining roots to be

$$\pm 2 \sin \frac{(2i+1)\pi}{4n} \qquad (i = 0, 1, 2, \ldots, n-1). \tag{2.11}$$

Finally, in our analysis of the roots of $F_n^*(x) = 0$ and $L_n^*(x) = 0$, we find from [1] that the $2n$ roots of

$$\Phi_{2n}(x) = 0 \tag{2.12}$$

are $\pm 2 \sin \dfrac{2i\pi}{2n+1} = \pm 2 \sin\left(\pi - \dfrac{2i\pi}{2n+1}\right)$, $i = 1, 2, \ldots, n$, that is, after manipulation,

$$\pm 2 \sin \frac{i\pi}{2n+1} \qquad (i = 1, 2, \ldots, n). \tag{2.13}$$

The roots of $F_{2n}^*(x) = 0$, $F_{2n+1}^*(x) = 0$, $L_{2n}^*(x) = 0$, and $L_{2n+1}^*(x) = 0$ are, respectively, those given in (1.5), (2.9), (2.11), and (2.13). See also [8]. It must be noted that the $2n - 2$ roots in (1.5) relate to $\dfrac{A_{4n}(x)}{x} = 0$ in (1.2), so $A_{4n}(x) = 0 = F_{2n}^*(x)$ in (2.4) has $(2n-2) + 1 = 2n - 1$ roots, one of these roots being $x = 0$. Also note the zero root associated with (2.7).

Verification of the Hoggatt-Bicknell roots is thus achieved by complex numbers in conjunction with the properties of the polynomials $A_n(x)$.

3. The Polynomials $A_{2n+1}(x)$

So far, the odd-subscript polynomials $A_{2n+1}(x)$ of degree n have not been featured. As mentioned in [4, pp. 245, 249],

$$A_{2n+1}(x) = \overline{f}_n(x) \tag{3.1}$$

in the notation of [1], where

$$f_n(x) = A_{2n+2}(x) - A_{2n}(x) = (-1)^n \overline{f}_n(-x) = (-1)^n A_{2n+1}(-x). \tag{3.2}$$

For instance,

$$\begin{aligned} f_5(x) &= A_{12}(x) - A_{10}(x) = x^5 - 4x^3 + 3x - (x^4 - 3x^2 + 1) \\ &= -(-x^5 + x^4 + 4x^3 - 3x^2 - 3x + 1) = -A_{11}(-x) \\ &= (-1)^5 \overline{f}_5(-x). \end{aligned}$$

Using the information given in [1] for the n roots of $\overline{f}_n(x) = 0$, we have that the n roots of

$$A_{2n+1}(x) = 0 \tag{3.3}$$

are

$$2 \cos \frac{2i\pi}{2n+1} \qquad (i = 1, 2, \ldots, n). \tag{3.4}$$

Thus, the two roots of $A_5(x) = \overline{f}_2(x) = x^2 + x - 1 = 0$ are

$$2 \cos \frac{2\pi}{5},\ 2 \cos \frac{4\pi}{5} \left(= -2 \cos \frac{\pi}{5}\right).$$

Following Legendre [6], Hancock [1] remarks that the equations

$$(-1)^n f_n(-x) = \overline{f}_n(x)$$

constitute a type of reciprocal equation obtained by substituting $z = x + \frac{1}{x}$ in $\frac{x^{2n+1} - 1}{x - 1} = 0$.

In [3, p. 55] it is shown that

$$A_{2n}(x) = \frac{s^{2n} - t^{2n}}{s^2 - t^2}, \tag{3.5}$$

where $s^2 = \frac{1}{2}(x + \sqrt{x^2 - 4})$ and $t^2 = \frac{1}{2}(x - \sqrt{x^2 - 4})$. Then

$$\begin{aligned} A_{4n+2}(x) &= A^2_{2n+2}(x) - A^2_{2n}(x) && \text{on using (3.5)} \\ &= (A_{2n+2}(x) - A_{2n}(x))(A_{2n+2}(x) + A_{2n}(x)) \\ &= f_n(x)\overline{f}_n(x) && \text{(Hancock [1])} \\ &= f_n(x)A_{2n+1}(x) && \text{by (3.1),} \end{aligned}$$

where we note as in [4, p. 248] that our $A_{2n}(x)$ is Hancock's $A_{n-1}(x)$. Thus,

$$f_n(x) = A_{4n+2}(x)/A_{2n+1}(x), \tag{3.6}$$

so that the $f_n(x)$ are expressible in terms of proper divisors. As an example, $A_{18}(x) = w_6(x)w_{18}(x)A_9(x)$, i.e.,

$$f_4(x) = A_{10}(x) - A_8(x) = w_6(x)w_{18}(x) = (x - 1)(x^3 - 3x - 1). \tag{3.7}$$

4. Concluding Comments

(a) The $2n$ roots of each equation

$$A_{4n+2}(x) = (-1)^n \sec\frac{2i\pi}{2n+1} \qquad (i = 1, 2, \ldots, n) \tag{4.1}$$

are shown in [1] to be

$$\pm 2\sin\frac{2i\pi}{2n+1} \qquad (i = 1, 2, \ldots, n). \tag{4.2}$$

Combining (2.5), (2.7), (2.12), (2.13) (in the equivalent form), (4.1), and (4.2) we see that

$$\frac{L^*_{2n+1}(x)}{ix} = 0 \text{ and } F^*_{2n+1}(x) - \sec\frac{2k\pi}{2n+1} = 0 \qquad (k = 1, 2, \ldots, n) \tag{4.3}$$

of degree $2n$, for a given n and a given value of k have the roots

$$\pm 2\sin\frac{2k\pi}{2n+1} \qquad (k = 1, 2, \ldots, n)$$

in common. For example, if $n = 2$ we find that $\pm 2 \sin \alpha\left(\alpha = \frac{2\pi}{5}, \frac{4\pi}{5}\right)$ are roots of $\dfrac{L^*_5(x)}{ix} = 0$ and $F^*_5(x) - \sec\alpha = 0$.

(b) It is observed in [1] that the curves

$$y = f_n(x) = A_{2n+2}(x) - A_{2n}(x) \quad [(3.2)] \qquad (n = 1, 2, \ldots)$$

all pass through the point with coordinates (2, 1), and through one or the other of the points (0, 1), (0, -1). Examples for easy checking are $y = f_4(x)$ given in (3.7), and $y = f_5(x)$ appearing after (3.2).

(c) Mention must finally be made of the very recent article by Kimberling [5] on cyclotomic polynomials which impinges on some of the content herein. Among other matters, one may compare Table 2 of [5] with Table 1 of [2].

If the irreducible divisors of the Fibonacci polynomials $F_n(x)$ given by (2.1) are represented by $\mathcal{F}_d(x)$ where $d|n$, then by [5, p. 114],

$$F_n(x) = \prod_{d|n} \mathcal{F}_d(x). \tag{4.4}$$

Allowing x to be replaced by ix in the polynomials $\mathcal{F}_n(x)$ occurring in Kimberling's Table 2, and writing the polynomial corresponding to $\mathcal{F}_n(x)$ as $\mathcal{F}_n^*(x)$, we find using [8] that

$$\mathcal{F}_p^*(x) = F_p^*(x) \qquad p \text{ prime} \tag{4.5}$$

$$\mathcal{F}_{2n}^*(x) = (-1)^{\frac{1}{4}\phi(4n)} w_{4n}(x) \qquad (n > 1), \tag{4.6}$$

where $\phi(n)$ is Euler's function and, by [4],

$$\text{deg. } w_n(x) = \frac{1}{2}\phi(n). \tag{4.7}$$

While the proof of (4.5) is straightforward, that of (4.6) requires some amplification. Now

$$F_{2n}^*(x) = (-1)^{n-1} i A_{4n}(x) \qquad \text{which is (2.4)}$$

$$\prod_{d|2n} \mathcal{F}_d^*(x) = (-1)^{n-1} ix \prod_{d|4n} w_d(x) \quad \text{by (4.4) amended and [4, p. 244]}$$

$$\prod_{d|2n} \mathcal{F}_d^*(x) = (-1)^{n-1} i \prod_{d|4n} w_d(x) \qquad n \geq 1 \text{ since } w_4(x) = x. \tag{4.8}$$

Apart from the sign (+ or -), the highest factor $\mathcal{F}_{2n}^*(x)$ on the left-hand side of (4.8) must equal the highest factor $w_{4n}(x)$ on the right-hand side of (4.8). This sign must, on the authority of (4.7), be

$$i^{\frac{1}{2}\phi(4n)} = (-1)^{\frac{1}{4}\phi(4n)},$$

whence (4.6) follows.

For example,

$$F_6^* = i(x^4 - 4x^3 + 3x) = ix(x^2 - 1)(x^2 - 3) = \mathcal{F}_2^*(x)(-\mathcal{F}_3^*(x))(-\mathcal{F}_6^*(x))$$

$$= iA_{12} = ix(x + 1)(x - 1)(x^2 - 3) = iw_4(x)w_3(x)w_6(x)w_{12}(x),$$

whence

$$\mathcal{F}_6^*(x) = -w_{12}(x) = (-1)^{\frac{1}{2}\phi(12)} w_{12}(x).$$

Kimberling's article opens up many ideas which we do not pursue here.

This concludes the linking together of material from several sources. Consideration of the polynomials $A_n(x)$ does indeed enable us to encompass a wide range of results.

References

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HOGGATT READING ROOM DEDICATION

On April 30, 1982, the Department of Mathematics at San Jose State University dedicated the

VERNER E. HOGGATT, JR. READING ROOM.

The room, opposite the offices of the Department of Mathematics, houses a splendid research library and various mathematical memorabilia. At the ceremony, Dean L. H. Lange of the School of Sciences talked of his long association with Professor Hoggatt and about Fibonacci numbers. A reception followed for faculty members and guests. Among the guests were various friends and associates of Professor Hoggatt, a number of whom are active in carrying on the work that Professor Hoggatt started with *The Fibonacci Quarterly*. Mrs. Hoggatt and her daughters attended the dedication ceremony, and Mrs. Hoggatt was presented with a portrait of her late husband.