ANALYSIS OF A BETTING SYSTEM

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1. Introduction

A friend of ours, on hearing about a "new" system for betting in roulette, did some initial investigating with pencil and paper, thought it looked good, and proceeded to try it out in Las Vegas. With a set goal and a capital he was willing to risk, he played the system religiously ... and won! This was the incentive for our more thorough investigation of the system. The outcome of the investigation may be guessed in advance; if not from a mathematical standpoint, surely from the facts that:

1. we have decided to publish the findings, and
2. neither of us is yet wealthy.

Since roulette is a game of (presumably) independent trials and since the house holds an edge on each trial, it is a foregone conclusion (see, for example, [1]) that there can be no betting scheme which gives the bettor a positive expectation. Nonetheless, there is a certain enticement to a scheme which is designed for use in a nearly "even" game of independent trials and which promises, by its nature, to leave the bettor ahead by a certain amount after the completion of a little routine which seems unavoidably destined for completion. Betting on red or black in roulette (probability of success with an American wheel is 18/38 since there are 18 red numbers, 18 black numbers, and, yes, two green numbers—0 and 00) provides the nearly "even" game. The scheme for betting in the game begins with a prechosen but arbitrary sequence of numbers $b_1, b_2, \ldots, b_n$, which we shall call the *betting sequence*. The algorithm to be followed is then:

1. (Make bet b) $b = b_1 + b_n$ if $n \geq 2$. $b = b_1$ if $n = 1$.
2. (Decrease betting sequence after a win) If win, then
   - 2.1. (Scratch outer numbers) Delete the values b_1 and b_n from the betting sequence.
   - 2.2. If sequence is exhausted, then halt. (Completion of a *betting cycle*)
   - 2.3. Decrease n by 2.
   - 2.4. (Relabel sequence numbers) Renumber remaining betting sequence to $b_1, \ldots, b_n$.

3. (Increase betting sequence after a loss) If lose, then
    3.1. Increase n by 1.
    3.2. (Attach current bet to sequence) Set $b_n = b$.
4. Repeat, starting at step 1.

As an example, suppose a bettor begins with the sequence 1, 2, 3, 4, 5. His first bet would be 6 units (1 + 5). If he wins that bet, his betting sequence becomes 2, 3, 4, and his next bet would be 6 units again (2 + 4). Given a loss of this second bet, his betting sequence would become 2, 3, 4, 6, and 8 units would be bet next. A complete betting cycle is illustrated below:

Trial No.	Betting Sequence	Bet	Outcome	Financial Status
1	1, 2, 3, 4, 5	6	Win	+6
2	2, 3, 4	6	Lose	Even
3	2, 3, 4, 6	8	Lose	-8
4	2, 3, 4, 6, 8	10	Win	+2
5	3, 4, 6	9	Win	+11
6	4	4	Lose	+7
7	4, 4	8	Win	+15

cycle complete (betting sequence exhausted)

Now the invitation to wealth is clear. With a nearly even chance of winning any bet and with the system scratching two numbers from the betting sequence on every win while adding only one number to the sequence on a loss, how can we fail eventually to exhaust the betting sequence? And sequence exhaustion beings with it a reward equal in monetary units to the sum of the numbers in the original betting sequence (easily proved). The only hitch in this otherwise wonderful plan is that there may come a time when we cannot carry a betting cycle through to completion simply because we do not have the resources to do so; i.e., we cannot cover the bet required by the system. (House limits on bets may also impose on our scheme, but these are not considered here.)

It turns out that this system is an old one called either *Labouchere* or *the cancellation system*. Mention is made of it (in a dismissing way) in the writings of professional gamblers (see [4], [6], [7], and [8]) and (in a promotional way) in one book [5], where the author claims to have won $163,000 using an anti-Labouchere system (turn around the win and lose actions) in a French casino in 1966.

Here we investigate Labouchere by first probing (in Section 2) a system which is somewhat similar to Labouchere, but more amenable to mathematical analysis. This gives a forecast of results to come. Next (Section 3) we look at Labouchere in a setting where there is no limit on the bettor's capital. Here the probabilities of cycle completion become clear. Finally, in Section 4, we simulate (mathemetical analysis seems very difficult) various situations under which Labouchere is applied with finite working capital. The intent is to display how the control of certain parameters (initial capital, goal, length of initial betting sequence, size and order of values in the original sequence) can impact the outcome statistics (frequency of goal achievement, mean bet size, mean number of bets to a win, mean earnings).

2. Analysis of a Simpler Scheme

Consider for a moment the popular *double-up* or *Martingale* betting system wherein the bettor doubles his wager after each loss and returns to his original bet after each win. This can be considered somewhat close to a Labouchere scheme by viewing the Martingale bettor as starting with a single number in his betting sequence, adding to the sequence any bet that he loses, betting the sum of the whole sequence, and deleting the whole sequence after any win. Thus, any win completes a betting cycle.

What "control" does the Martingale bettor have over his fortunes? Suppose the probability of success on any trial is p and let $q = 1 - p$. For simplicity, we let the gambler's initial capital be

$$C_0 = (2^k - 1)b$$

for some positive integers b and k, where b is the amount to be bet initially. We shall also say that the gambler's profit goal is G and, again for simplicity, set $G = mb$, where $0 < m < 2^k$. Under this arrangement, the bettor must experience m successful trials (complete betting cycles) to achieve his profit goal, while k consecutive losses will ruin him (i.e., leave him with insufficient resources to continue with the Martingale scheme). Thus,

$$\text{Prob[achieve } G] = (1 - q^k)^m.$$

We note immediately that, given the same capital, the greedier gambler (the one with a larger profit goal) has a smaller probability of achieving his objective, but that the amount by which this probability diminishes with increased ambition depends on the bettor's initial capital.

Now assume that the gambler will achieve his profit goal $G = mb$, and let $X_1, X_2, \ldots, X_m$ be random variables whose values are determined by the number of trials needed to complete cycle 1, cycle 2, ..., cycle m, respectively. Then the expected number of trials to achieve G (given that G will be achieved) is

$$E[X_1 + X_2 + \cdots + X_m] = \frac{1}{1 - q^k} \sum_{j=1}^{m} \sum_{i=1}^{k} i\, pq^{i-1} = \frac{mp}{1 - q^k} \sum_{i=1}^{k} i\, q^{i-1}.$$

To get a feeling for these numbers, we present some examples in Table 1 where we assume that $p = 18/38$ (as in roulette) and that in each case shown the initial amount wagered is 10 units (i.e., $b = 10$).

The main observations that we wish to make from Table 1 are that under a Martingale system a bettor has the following "controls":

1. He can adjust his probability of achievement of the profit goal, G, by adjusting his goal-to-initial capital ratio (G/C_0). This apparent dependence is shown dramatically in Figure 1, a plot of Table 1 data.

TABLE 1

Expected Results Using a Martingale Betting Scheme with Initial Bet \$10 and $p = 18/38$

m	k	Profit Goal (G) $G = 10m$	Initial Capital (C_0) $C_0 = (2^k - 1)10$	Prob[achieve G] $P_G = (1 - q^k)^m$	Expected Number of Trials to achieve G	Expected Earnings per Play $GP_G - C_0(1 - P_G)$
5	3	50	70	.455	7.995	-15.40
4	3	40	70	.532	6.396	-11.48
3	3	30	70	.623	4.797	-7.70
10	5	100	310	.662	21.350	-38.58
5	4	50	150	.671	9.612	-15.80
9	5	90	310	.690	19.217	-34.00
8	5	80	310	.719	17.082	-29.59
4	4	40	150	.727	7.690	-11.87
2	3	20	70	.730	3.198	-4.30
7	5	70	310	.749	14.947	-25.38
6	5	60	310	.781	12.811	-21.03
3	4	30	150	.787	5.767	-8.34
5	5	50	310	.814	10.676	-16.96
4	5	40	310	.848	8.541	-13.20
2	4	20	150	.852	3.845	-5.16
1	3	10	70	.854	1.599	-1.68
3	5	30	310	.883	6.406	-9.78
5	6	50	630	.898	11.348	-19.36
4	6	40	630	.918	9.078	-14.94
2	5	20	310	.921	4.270	-6.07
1	4	10	150	.923	1.922	-2.32
3	6	30	630	.938	6.809	-10.92
2	6	20	630	.958	4.539	-7.30
1	5	10	310	.960	2.135	-2.80
1	6	10	630	.979	2.270	-3.44

2. He can adjust his expected time for achieving his goal. The general rule here seems to be that a need for more cycles to achieve the goal increases expected achievement time as does having a larger initial capital. That is, increasing either k or m increases the expected number of trials for gaining G.
3. The more a gambler is willing to risk (C_0) and the greedier he is (G), the larger his expected loss.

We shall see shortly that the Labouchere bettor has the same sorts of controls over his fortunes, but that his setting provides for more controls, in that he also has a choice of betting sequence. This is the real complicating factor in the analysis of Labouchere. We turn now to some of the mechanics of the cancellation scheme before going into the full simulation of practical situations.

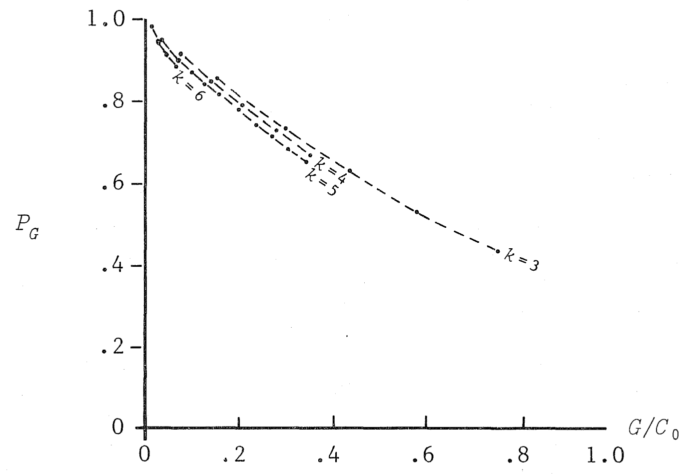

FIGURE 1

A plot of the Martingale results in Table 1 showing the relationship between probability of goal achievement and goal-to-initial-capital ratio. Dashes connect points with same initial capital.

3. The Case of Infinite Capital

Throughout this section we shall assume that our Labouchere bettor really does not care what sorts of temporary losses he incurs, for he has enough money to cover any loss. His only real concern is how long it will take him to recover the loss by completing his betting cycle. He asks then for the probability that he will complete a cycle in t or fewer trials.

Suppose the initial betting sequence consists of n numbers. Let w represent the number of wins in t trials and let $\ell = t - w$ represent the number of losses. In order for a betting cycle to be completed on trial t we must have

$$2w \geq \ell + n \quad \text{and} \quad 2(w - 1) < \ell + n. \tag{1}$$

That is, since two numbers are deleted from the sequence with each win and only one number added for each loss, the first inequality gives a condition for sequence exhaustion and the second assures that the sequence was not exhausted on the $(t-1)$st trial. Together, these yield

$$\frac{1}{3}(t + n) \leq w < \frac{1}{3}(t + n + 2). \tag{2}$$

Because w is an integer and the two extremes of this inequality differ by only 2/3, there can be at most one solution w for given t and n. In fact, since t and w are also integers, the only situation in which no such w exists will be that for which

$$\frac{1}{3}(t + n) = m + \frac{1}{3},$$

where m is an integer; i.e.,

$$t \equiv 2n + 1 \pmod 3.$$

Hence, if $t = 3k + i$ and $n = 3h + j$, where $0 \le i,\ j \le 2$, then

$$w = \begin{cases} k + h, & \text{if } i = j = 0 \\ k + h + 1, & \text{otherwise, unless } i + j \equiv 1 \pmod 3, \\ & \text{where there is no solution.} \end{cases}$$

From the way we set up our conditions to find w, it is seen that not every permutation of w wins and $t - w$ losses will result in cycle completion on trial t. (Some will dictate earlier sequence exhaustion.) However, every t-trial cycle completion with an initial betting sequence of n numbers will involve exactly w wins where w is determined as above.

Our question now becomes: How many permutations of w wins in t trials result in cycle completion on trial t? To address this, we make our setting more definite, and note simply that other settings are similar. We take the case where there are five numbers in the original betting sequence ($n = 5$). In this case our analysis above shows that it requires exactly

$$w = k + 2$$

wins to complete a betting cycle in $t = 3k + i$ trials, providing $i = 0$ or 1. It is impossible to complete a cycle in t trials if $i = 2$.

Figure 2 shows a graph in the $w\ell$-plane of the inequalities (1), which here become

$$2w \ge \ell + 5 \quad \text{and} \quad 2(w - 1) < \ell + 5.$$

The lines $t = w + \ell$ are shown at various levels. We consider a random walk on this graph where each loss corresponds to a positive unit step vertically and each win corresponds to a positive unit step horizontally. Beginning at the origin, we hope to follow the determined path into the region described by the inequalities, since this corresponds to completing a cycle. Hence, we call this region the *completion zone*.

For the purpose of restating our question in this new context, let us say that a path in our random walk from the origin to some point (a, b) is *permissible* if it never enters the completion zone before reaching (a, b). Then our question asks how many different permissible paths lead to the point

$$(k + 2,\ 2k + i - 2),\ k \ge 1,\ i = 0 \text{ or } 1.$$

Now a recursion formula that answers the question is easily derived from noticing that any path leading to (a, b) in this random walk must have as its last step either the step from $(a - 1, b)$ to (a, b) or from $(a, b - 1)$ to (a, b). So denoting the number of permissible paths to (a, b) by $N(a, b)$, we have

$$N(a, b) = N(a - 1, b) + N(a, b - 1).$$

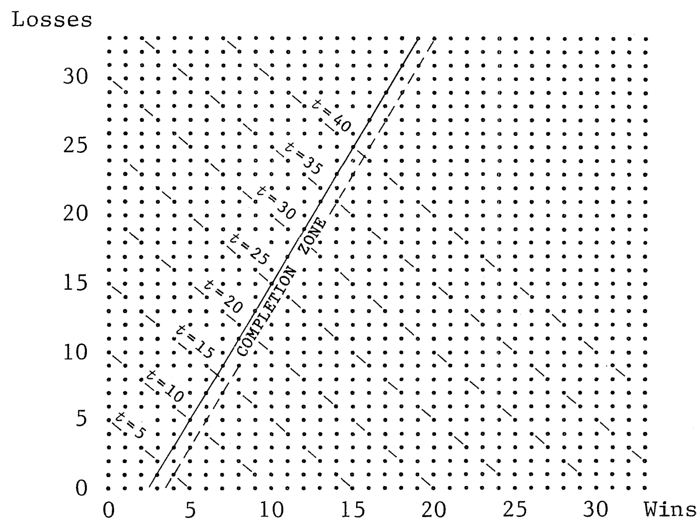

FIGURE 2

The Random Walk Setting with n = 5

This formula is, of course, subject to the provision that neither $(a - 1, b)$ nor $(a, b - 1)$ is in the completion zone, for then the path to (a, b) would not be permissible. So, for example, if $(a, b - 1)$ is in the completion zone, then $N(a, b) = N(a - 1, b)$.

Putting this in terms of t and w rather then w and ℓ and denoting the number of permissible ways to achieve w wins in t trials by $\left\{ t \atop w \right\}$, our basic formula becomes

$$\left\{ t \atop w \right\} = \left\{ t - 1 \atop w - 1 \right\} + \left\{ t - 1 \atop w \right\}$$

with the same provision that if either $t - 1$ and $w - 1$ or $t - 1$ and w determine a point in the completion zone, then the corresponding number is not added in the formula. These numbers clearly act somewhat like binomial coefficients and, in fact, we get a modification of Pascal's triangle as shown in Figure 3. There the circled items represent numbers corresponding to points in the completion zone and, consequently, are not added in the derivation of the succeeding row. Now we have, for example, that $\left\{ 9 \atop 4 \right\} = 83$; that is, there are 83 permissible paths to the point (4, 5) in Figure 2. This, in turn, is equivalent to saying that there are 83 sequences consisting of four wins and five losses which lead to completion of the betting cycle in exactly nine trials.

t													
0	1												
1	1	1											
2	1	2	1										
3	1	3	3	1									
4	1	4	6	3	0								
5	1	5	10	6	0	0							
6	1	6	15	16	6	0	0						
7	1	7	21	31	16	0	0	0					
8	1	8	28	52	31	0	0	0	0				
9	1	9	36	80	83	31	0	0	0	0			
10	1	10	45	116	163	83	0	0	0	0	0		
11	1	11	55	161	279	163	0	0	0	0	0	0	
12	1	12	66	216	440	442	163	0	0	0	0	0	0

FIGURE 3. $\left\{ {t \atop w} \right\}$

These counts of the number of ways to complete a cycle in exactly t trials can be written explicitly in terms of binomial coefficients and, somewhat more neatly, in terms of binomial coefficients and the analogous numbers associated with a three-number initial betting sequence. We sketch the derivation of this latter expression in the appendix to this paper.

Given these numbers, we have essentially answered the question posed by the infinitely wealthy gambler at the beginning of this section. For if the probability of a win on any turn is p, then the probability of completing a cycle on or before the tth trial (still assuming $n = 5$) is found by adding terms of the form

$$\left\{ {3k + i \atop k + 2} \right\} p^{k+2}(1 - p)^{2x+i-2}$$

with $i = 0$ or 1 and k ranging from 1 to $\left[\frac{t}{3}\right]$ with the restriction that

$$3k + i \leq t.$$

Some of these numbers are given in Table 2 below under the assumption that p is, again, 18/38.

The mean of such a distribution of the number of trials (bets) needed to complete a cycle with an initial betting sequence of length n can be found without too much trouble. We let X_i, for $i = 1, 2, 3, \ldots$, be a random variable which takes the value -2 if bet i is won and +1 if bet i is lost. Then after t bets we see that $S_t = X_1 + X_2 + \cdots + X_t$ gives the change in length

TABLE 2

Completion Probabilities with a Five-Member Initial Sequence and p = 18/38

t	Probability of Completion on Trial t	Probability of Completion within t Trials
3	.1063	.1063
4	.1678	.2741
6	.0837	.3578
7	.1174	.4752
9	.0567	.5319
10	.0799	.6119
12	.0391	.6510
13	.0559	.7069
15	.0278	.7347
16	.0401	.7748
18	.0202	.7950
19	.0295	.8246
21	.0151	.8396
22	.0221	.8617

of the betting sequence from its original length. For any t we have

$$E[S_t] = E[X_1]t,$$

since the X_i's are identically distributed and independent. Since

$$E[X_1] = 1 - 3p,$$

where p is the probability of success on any trial, then

$$E[S_t] = (1 - 3p)t.$$

We note that, in terms of wins and losses, if we combine the conditions

$$w + \ell = t$$

$$-2w + \ell = (1 - 3p)t,$$

we get the line

$$\ell = \frac{1 - p}{p} w .$$

Plotting this line of expected results on a graph like that in Figure 2 and extending it to meet the completion zone, we can get an idea of the expected number of trials to complete a cycle by computing the point of intersection with the completion zone boundary line. We get

$$E[T] = \frac{1}{3p - 1}n,$$

where T is the random variable whose value is the number of trials in the completion of a cycle. In fact, this is a geometric version of Wald's identity (see [3]) which relates $E[S_T]$ and $E[T]$.

Of course, this analysis addresses only the number of bets needed to complete a cycle and, like the infinitely wealthy gambler, ignores any consideration of the money involved in completing a cycle. In the next section, our gambler has finite capital and the difficult questions of financial impact of parameter adjustment become paramount.

4. A More Realistic Setting

Consider the following two betting cycles, each of which is completed on the 10th trial:

Trial	Bet Sequence	Bet	Outcome	Financial Status
1	1, 2, 3, 4, 5	6	Win	+6
2	2, 3, 4	6	Win	+12
3	3	3	Lose	+9
4	3, 3	6	Lose	+3
5	3, 3, 6	9	Win	+12
6	3	3	Lose	+9
7	3, 3	6	Lose	+3
8	3, 3, 6	9	Win	+12
9	3	3	Lose	+9
10	3, 3	6	Win	+15
	Exhausted			
1	1, 2, 3, 4, 5	6	Win	+6
2	2, 3, 4	6	Lose	Even
3	2, 3, 4, 6	8	Win	+8
4	3, 4	7	Lose	+1
5	3, 4, 7	10	Lose	-9
6	3, 4, 7, 10	13	Lose	-22
7	3, 4, 7, 10, 13	16	Lose	-38
8	3, 4, 7, 10, 13, 16	19	Win	-19
9	4, 7, 10, 13	17	Win	-2
10	7, 10	17	Win	+15
	Exhausted			

Notice that in each cycle there occurred five wins and five losses, as expected for a completion on trial ten. However, the money required of the bettor greatly differed between the two cycles. In the first, the bettor needed only enough money to cover his first bet (6 units). From there on he was always "ahead of the game." But in the second cycle, the bettor needed to have an initial capital of at least 57 units in order to be able to bet 19 units on the eighth trial, while being 38 units behind. So we see that the arrangement of the win-loss sequence in a cycle of fixed length can have great impact on the amount of money needed to survive the cycle. It is this dependence of monetary needs on both the bet sequence and the sequence of wins and losses that drives us to the computer in an effort to understand generally what can be expected from various situations. Using a random number generator we have simulated (naturally, a Monte Carlo simulation) a tournament of Labouchere gamblers. The rules of the tournament were:

1. Each player begins with \$500 and tries to realize a profit of \$60.
2. Each player must strictly follow a given Labouchere scheme until he either earns the \$60 profit or cannot meet the bet level necessary to continue playing. At such time, he is given another \$500 and begins another play of his system.
3. All players gamble simultaneously at the same American roulette wheel until they have completed at least 2,000 plays and at least 62,500 spins of the wheel.

The 24 simulated players who competed in this tournament (which took under 2 minutes of computer time) had various ideas about what constitutes a good betting sequence. The following fairly well characterize the two extremes in these ideas:

Claim of Gambler A: If I structure my sequence so that generally my bets are quite small relative to my capital, then chances are that I'll have sufficient capital to survive most streaks of misfortune.

Claim of Gambler U: I'll use a sequence which is short and requires only one cycle completion to achieve the profit goal. This way, on any given play I probably won't be around long enough to run into a disastrous win-loss sequence. Besides, with my bets being fairly large, chances are that not being able to cover a bet still leaves me with substantial capital (i.e., a ruin is not so bad).

The 24 simulated gamblers and a host of simulated officials gathered around the simulated wheel of fortune and watched it spin more than a quarter of a million times until the final player had completed his 2,000 plays. (A required 254,661 bets to complete 2,000 plays.) The results, as reported in Table 3 in ascending order of goal achievement rate, tend to support the notions of gambler U, up to a point. We do see that, initially, the sequences with fewer cycles needed to achieve the goal yield better returns in terms of both achievement percentage and mean earnings per play. However, toward the bottom on Table 3 some leanings toward player A's ideas can be noted. Where, under player U's philosophy, we would have expected his ultimately short sequence to have done better then the sequences of players V, W, or X, we see instead that, apparently on occasion, player U's bets built up a little too quickly for his \$500 capital to withstand, while the sequences of players V, W, and X allowed for more moderate build-up and a better achievement percentage. Compare the mean bet size of players R, S, T, U, V, W, X to see this. Note, however, that the six 10s of player S allowed for a moderate bet size, too, but also required substantially more bets to complete a winning cycle. Note, too, that while players W and X achieved the profit goal most frequently, player U was correct about mean earnings and fared better than anyone in that category.

TABLE 3

Simulation Results for a Minimum of 2,000 Plays and 62,500 Bets at an American Roulette Wheel Using Various Betting Sequences to Attempt to Achieve a Profit Goal of $60 from an Initial Capital of $500

Player	Initial Bet Sequence	% of Plays Goal Achieved	Mean Bet Size	Mean # of Bets to Win	Mean Earnings per Play	# of Plays	# of Cycles to Achieve Goal
A	1,1,1,1,1,1	74.6	10.26	136.7	-64.86	2000	10
B	1,2,3	76.6	14.76	71.1	-54.16	2000	10
C	3,2,1	76.6	14.87	71.1	-53.97	2000	10
D	2,2,2	76.8	14.97	71.0	-52.32	2000	10
E	5,1	78.2	19.36	43.8	-45.65	2000	10
F	4,2	78.2	19.62	43.9	-45.04	2000	10
G	2,4	78.4	19.95	43.9	-44.47	2000	10
H	3,3	78.6	19.87	43.9	-43.76	2000	10
I	5,4,1,2,3	79.0	20.55	41.6	-42.71	2000	4
J	1,5	79.0	19.81	44.1	-41.68	2000	10
K	1,2,3,4,5	79.3	20.30	42.0	-41.89	2000	4
L	5,4,3,2,1	79.4	20.67	41.7	-41.26	2000	4
M	6	79.7	23.99	30.6	-38.78	2040	10
N	7,5,3	80.4	27.44	25.8	-35.81	2368	4
O	3,5,7	80.4	27.58	25.8	-35.55	2362	4
P	5,5,5	80.5	27.57	25.8	-35.16	2367	4
Q	5,10	80.8	36.42	15.3	-32.59	3809	4
R	10,5	81.0	36.08	15.5	-31.57	3802	4
S	10,10,10,10,10,10	82.2	46.73	10.2	-26.77	5470	1
T	50,10	82.2	87.43	2.7	-12.42	17018	1
U	60	83.2	115.36	1.8	-10.52	30117	1
V	10,50	83.5	93.19	2.6	-13.84	16380	1
W	10,20,30	83.6	68.55	4.8	-18.35	11160	1
X	30,20,10	84.3	68.66	5.0	-17.52	11143	1

To a good extent these results reflect what is generally the case in the classical setting where a constant amount is wagered on each trial. In that situation an increase in bet size (with initial capital held constant) brings a decrease in probability of ruin for a player whose probability of success on any trial is less than 1/2 (see [2, p. 347]). This principle needs modification under Labouchere only where bet sizes tend to grow too rapidly for underlying capital.

Table 4 gives results of another simulation which was run as a study of the effects of initial capital on relative frequency of goal achievement. Here 15 players stood around the same wheel (actually playing along with the 24 players in the first simulation) betting the same Labouchere system (bet sequence 1, 2, 3, 4, 5), aiming for a $60 profit, but starting with different capital amounts. The effects can be noted to be much like those under a Martingale scheme by comparing Figure 1 in Section 2 with Figure 4, which gives a visual presentation of Table 4 entries. Again, the wealthy, unambitious gambler has a high likelihood of goal achievement, but a worse expectation since he loses so much in the infrequent disasters he encounters.

TABLE 4

Simulation Results for a Minimum of 2,000 Plays and a Minimum of 62,500 Bets at an American Roulette Wheel in an Effort to Gain a Profit Goal of $60 Using Betting Sequence 1,2,3,4,5 from Various Initial Capital Values

Initial Capital	Percentage of Plays Goal Achieved	Mean Bet Size	Mean Number of Bets to a Win	Mean Earnings per Play	Number of Plays
60	33.3	10.97	30.3	-11.67	2809
80	39.6	11.82	32.1	-15.05	2440
100	45.0	12.47	33.3	-17.53	2217
120	49.6	13.17	34.4	-20.18	2062
150	54.9	14.11	36.0	-23.55	2000
180	59.5	14.97	37.0	-25.70	2000
240	65.7	16.20	38.6	-30.35	2000
300	70.6	17.53	39.8	-32.28	2000
500	79.3	20.30	42.0	-41.89	2000
600	81.6	21.58	42.5	-46.44	2000
1200	88.7	27.30	45.6	-63.77	2000
1800	91.2	30.98	46.7	-83.17	2000
2400	92.5	33.79	47.1	-99.61	2000
3000	93.8	36.60	47.9	-104.18	2000
6000	96.1	44.74	48.6	-148.92	2000

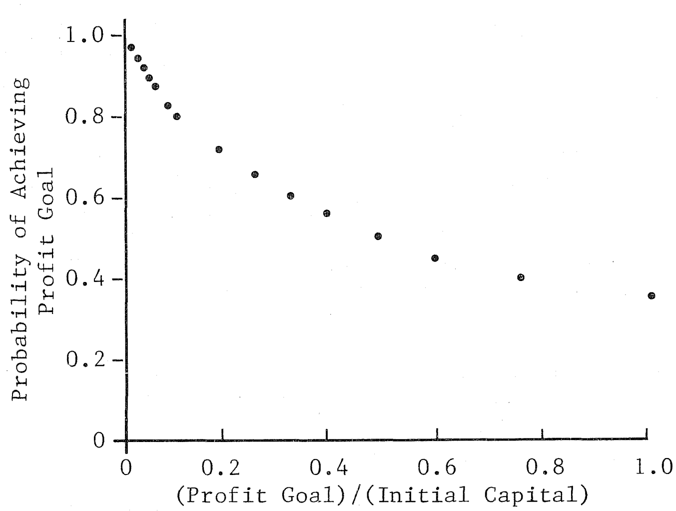

FIGURE 4. A Summary of Entries in Table 4

Just to see what sort of capital it would take to make it successfully through all 2,000 plays under the setting of Table 4, we simulated a gambler with \$9,000,000 initial capital rather modestly seeking the \$60 profit goal. He did win all 2,000 plays of the system, but needed to place some HUGE bets from time to time in order to complete a betting cycle. In five of the 8,000 cycles which he completed he was forced to lay down bets exceeding \$100,000. His moment of most concern occurred when a bet of \$751,440 was demanded by the system and his capital was down to \$6,926,517. This, of course, suggests that the gambler had to have a minimum of \$2,824,923 in working capital in order to survive all 2,000 plays. Consequently, without an unreasonably large capital relative to a given profit goal, we cannot expect to play a Labouchere scheme without occasional losses. It is, as with other schemes, possible to manipulate the probability of achieving the profit goal and the expected duration of the betting, but, alas, the losses more than cover the gains eventually. We are, as noted at the outset, victims of the house edge.

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Appendix

In this appendix we first investigate the number of ways in which a betting cycle can be completed in exactly t trials under the assumption that our initial betting sequence contains only three numbers. Having answered that question, we shall then apply the result to the now more familiar situation of a five-number initial sequence answering the same question in that setting. The result is easily generalized to fit any initial sequence.

Given a three-number initial sequence, let

$$C_j = \begin{cases} 1, \text{ if } j = 0 \text{ or } -1 \\ \text{the number of ways of completing a cycle in exactly } j \text{ trials,} \\ \text{if } j > 0. \end{cases}$$

In this new setting, the inequalities corresponding to (2) in Section 3 are

$$\frac{1}{3}(t + 3) \leq w < \frac{1}{3}(t + 5).$$

These, as before, give the conditions for completing a cycle. Consequently, in this setting a cycle can be completed only when $t \equiv 0$ or $2 \pmod 3$. This gives us $C_{3k+1} = 0$ for $k = 0, 1, 2, \ldots$.

With an argument like the one advanced in Section 3, we see that if $t = 3k + i$ where $i = -1$ or 0 and $k > 1$, exactly $k + 1$ wins are needed to complete a cycle in t trials. However, these wins must be distributed over the trials so that the cycle is not completed before trial t.

For example, exactly 10 wins are required to complete a cycle in 27 trials, but if 9 of these occurred in the first 24 trials, the cycle would have previously completed. Similarly, no cycle which completes in exactly 27 trials will have 8 wins in the first 21 trials, and so on. We are faced with the model shown in Figure 5,

Trials:

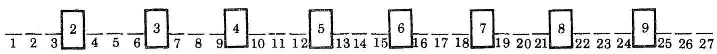

FIGURE 5

wherein we consider the number of ways to place 10 wins among the 26 trials, while respecting the barriers shown. The number on each barrier is meant as a strict upper bound to the number of wins which may fall to the left of the barrier.

To compute C_{27} we start with the observation that the last trial of any complete cycle must be a win. Consequently, we must determine in how many ways 9 wins may be appropriately distributed among the first 26 trials. First, $\binom{26}{9}$ gives the number of ways to do this distribution without regard to barriers. From this we first subtract the number of win-loss sequences in which the 9 wins occur in trials 1-24, $\binom{24}{9}$. Next we need to subtract out of of these $\binom{26}{9}-\binom{24}{9}$ remaining win-loss sequences those which do not respect the barrier at 21, and have not yet been subtracted; i.e., those with 8 wins in the first 21 trials and the 9th win in the 25th or 26th trial—$2\binom{21}{8}$.

Now we are left with $\binom{26}{9} - \binom{24}{9} - 2\binom{21}{8}$ win-loss sequences, all of which respect the rightmost two barriers of the figure. How many of these should be subtracted out for violating the restriction on the barrier at 18? Such a sequence would have 7 wins in the first 18 trials and two more wins in trials 22 to 26 respecting the barrier at 24. Notice this last condition of disstributing two wins appropriately among trials 22 to 26 with a fixed win in trial 27 is exactly the condition for completing a cycle in six trials (i.e., having a win in the 6th trial and distributing two more wins among trials 1-3). Consequently, the number of sequences to be subtracted out at this step is

$$C_6\binom{18}{7}.$$

Continuing this reasoning, we finally get

$$C_{27} = \binom{26}{9} - C_0\binom{24}{9} - C_3\binom{21}{8} - C_6\binom{18}{7} - C_9\binom{15}{6} - C_{12}\binom{12}{5}$$
$$- C_{15}\binom{9}{4} - C_{18}\binom{6}{3} - C_{21}\binom{3}{2}.$$

And a simple inductive argument gives us that

$$C_t = \begin{cases} 1, \text{ if } t = -1, 0 \\ 0, \text{ if } t = 3k + 1, k \geq 0 \\ \dbinom{3k + i - 1}{k} - \displaystyle\sum_{\ell=0}^{k-2} C_{3\ell+i}\dbinom{3(k - \ell - 1)}{k - \ell}, \text{ if } t = 3k + i,\ k \geq 1, \text{ and } i = 0 \text{ or } -1, \end{cases}$$

where we take a sum to be zero if its upper index limit is less than its lower index limit.

Returning to our example with the five-number initial betting sequence, an argument entirely like the preceding one leads us to

$$\left\{ 3k + i \atop k + 2 \right\} = \binom{3k + 2i - 2}{k + i} - \sum_{\ell=0}^{k+i-3} C_{3\ell+2i}\binom{3k + 3i - 5 - 3\ell}{k + i - \ell},$$

where $k \geq 1$, $i = 0$ or 1, and the C_j are the same numbers as before (arising in the three-number initial sequence case). Here, again, we take sums whose upper index limit is less than the lower index limit to be zero.

This, of course, is easily generalized to the case where the length of the initial betting sequence is arbitrary.

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